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On the vanishing ranges for the cohomology of finite groups of Lie type II

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Abstract. The computation of the cohomology for finite groups of Lie type in the describing characteristic is a challenging and difficult problem. In [BNP], the authors constructed an induction functor which takes modules over the finite group of Lie type, $G(\mathbb{F}_q)$, to modules for the ambient algebraic group G. In particular this functor when applied to the trivial module yields a natural G-filtration. This filtration was utilized in [BNP] to determine the first non-trivial cohomology class when the underlying root system is of type A_n or C_n. In this paper the authors extend these results toward locating the first non-trivial cohomology classes for the remaining finite groups of Lie type (i.e., the underlying root system is of type B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2) when the prime is larger than the Coxeter number.

1. Introduction

1.1. Let G be a simple algebraic group scheme over a field k of prime characteristic p which is defined and split over the prime field $\mathbb{F}_p$, and $F : G \to G$ denote the Frobenius map. The fixed points of the rth iterate of the Frobenius map, denoted $G(\mathbb{F}_q)$, is a finite Chevalley group where $\mathbb{F}_q$ denotes the finite field with p^r elements. An elusive problem of major interest has been to determine the cohomology ring $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$. Until recently, aside from small rank cases, it was not even known in which degree the first non-trivial cohomology class occurs.

This present paper is a sequel to [BNP] where we began investigating three related problems of increasing levels of difficulty:

(1.1.1) Determining Vanishing Ranges: Finding $D > 0$ such that the cohomology group $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < D$.

(1.1.2) Locating the First Non-Trivial Cohomology Class: Finding a D satisfying (1.1.1) such that

$$\mathrm{H}^D(G(\mathbb{F}_q), k) \neq 0.$$

A D satisfying this property will be called a *sharp bound*.

(1.1.3) Determining the Least Non-Trivial Cohomology: For a sharp D as in (1.1.2) compute $\mathrm{H}^D(G(\mathbb{F}_q), k)$.

Vanishing ranges (1.1.1) were found in earlier work of Quillen [Q], Friedlander [F] and Hiller [H] . Sharp bounds (1.1.2) were later found by Friedlander and Parshall for the Borel subgroup $B(\mathbb{F}_q)$ of the $GL_n(\mathbb{F}_q)$, and conjectured for the general linear group by Barbu [B]. A more detailed discussion of these results can be found in [BNP, Section 1.1].

In [BNP], for simple, simply connected G and primes p larger than the Coxeter number h, we proved that $\mathrm{H}^i(G(\mathbb{F}_{p^r}), k) = 0$ for $0 < i < r(p-2)$. This provided an answer to (1.1.1) and improved on Hiller's bounds [H]. For a group with underlying root system of type C_n, we demonstrated that $D = r(p-2)$ is in fact a sharp bound, answering (1.1.2). The first non-vanishing cohomology, as in (1.1.3), was also determined. For type A_n, questions (1.1.2) and (1.1.3) were also answered, where the $r > 1$ cases required the prime to be

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at least twice the Coxeter number. Our methods also yielded a proof of Barbu's Conjecture [B, Conjecture 4.11].

In this paper, we continue these investigations in two directions. First we consider the case that G is a group of adjoint type (as opposed to simply connected). For such G with $p > h$, when the root system is simply laced, one obtains a uniform sharp bound of $r(2p-3)$ answering (1.1.2) (cf. Corollary 3.3.1). The same uniform bound also holds for the adjoint versions of the twisted groups of types A, D, and E_6 when $p > h$.

We then consider the remaining types in the simply connected case. For G being simple, simply connected and having root system of type D_n with $p > h$, (1.1.2) and (1.1.3) are answered (cf. Theorem 4.5.2). For type E_n, (1.1.2) is answered for all primes $p > h$ (with the exceptions of $p = 17, 19$ for type E_6); cf. Theorems 5.1.3, 5.2.3, and 5.3.1.

The calculations for the non-simply-laced groups are considerably more complicated. For type B we answer (1.1.2) when $r = 1$ and $p > h$, see Theorem 6.7.1. Some discussion of the situation for types G_2 and F_4 is given in Sections 7 and 8 respectively. For $r = 1$ and $p > h$, we find improved answers to (1.1.1); cf. Theorem 7.5.1 and Theorem 8.1.1. Finding an answer to (1.1.2) and (1.1.3) continues to be elusive in these types although some further information towards answering these questions is obtained.

1.2. Notation. Throughout this paper, we will follow the notation and conventions given in the standard reference [Jan]. G will denote a simple, simply connected algebraic group scheme which is defined and split over the finite field $\mathbb{F}_p$ with p elements (except in Section 3.3 where G is assumed to be of adjoint type rather than simply connected). Throughout the paper let k be an algebraically closed field of characteristic p. For $r \geq 1$, let $G_r := \ker F^r$ be the rth Frobenius kernel of G and $G(\mathbb{F}_q)$ be the associated finite Chevalley group. Let T be a maximal split torus and Φ be the root system associated to (G, T). The positive (resp. negative) roots are Φ^+ (resp. Φ^-), and Δ is the set of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B. For a given root system of rank n, the simple roots will be denoted by $\alpha_1, \alpha_2, \dots, \alpha_n$ (via the the Bourbaki ordering of simple roots). For type B_n, α_n denotes the unique short simple root and for type C_n, α_n denotes the unique long simple root. The highest (positive) root will be denoted $\tilde{\alpha}$, and for root systems with multiple root lengths, the highest short root will be denoted α_0. Let W denote the Weyl group associated to Φ, and, for $w \in W$, let $\ell(w)$ denote the length of the element w (i.e., number of elements in a reduced expression for w).

Let $\mathbb{E}$ be the Euclidean space associated with Φ, and the inner product on $\mathbb{E}$ will be denoted by $\langle\ ,\ \rangle$. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$. The fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \dots, \omega_n$. Let $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$. For a weight $\lambda \in X(T)$, set $\lambda^* := -w_0\lambda$ where w_0 is the longest word in the Weyl group W. By $w \cdot \lambda := w(\lambda + \rho) - \rho$ we mean the "dot" action of W on $X(T)$, with ρ being the half-sum of the positive roots. For $\alpha \in \Delta$, $s_\alpha \in W$ denotes the reflection in the hyperplane determined by α.

For a G-module M, let $M^{(r)}$ be the module obtained by composing the underlying representation for M with F^r. Moreover, let M^* denote the dual module. For $\lambda \in X(T)_+$, let $H^0(\lambda) := \operatorname{ind}_B^G \lambda$ be the induced module and $V(\lambda) := H^0(\lambda^*)^*$ be the Weyl module of highest weight λ.

2. General Strategy and Techniques

2.1. We will employ the basic strategy used in [BNP] in addressing (1.1.1)-(1.1.3) which uses effective techniques developed by the authors which relate $\mathrm{H}^i(G(\mathbb{F}_q), k)$ to extensions over G via an induction functor $\mathcal{G}_r(-)$. When $\mathcal{G}_r(-)$ is applied to the trivial module k, $\mathcal{G}_r(k)$ has a filtration with factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ [BNP, Proposition 2.4.1]. The G-cohomology of these factors can be analyzed by using the Lyndon-Hochschild-Serre (LHS) spectral sequence involving the Frobenius kernel G_r (cf. [BNP, Section 3]). In particular for $r = 1$, we can apply the results of Kumar-Lauritzen-Thomsen [KLT] to determine the dimension of a cohomology group $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$, which can in turn be used to determine $\mathrm{H}^i(G(\mathbb{F}_{p^r}), k)$. The dimension formula involves the combinatorics of the well-studied Kostant Partition Function. This reduces the question of the vanishing of the finite group cohomology to a question involving the combinatorics of the underlying root system Φ.

For root systems of types A and C the relevant root system combinatorics was analyzed in [BNP, Sections 5-6]. In the cases of the other root systems (B, D, E, F, G) the combinatorics is much more involved and

we handle these remaining cases in Sections 4-8. In this section, for the convenience of the reader, we state the key results from [BNP] which will be used throughout this paper.

2.2. We first record here a formula for $-w\cdot 0$ that will be used at various times in the exposition [BNP, Observation 2.1]:

OBSERVATION 2.2.1. If $w \in W$ admits a reduced expression $w = s_{\beta_1}s_{\beta_2}\dots s_{\beta_m}$ with $\beta_i \in \Delta$ and $m = \ell(w)$, then
$$-w\cdot 0 = \beta_1 + s_{\beta_1}(\beta_2) + s_{\beta_1}s_{\beta_2}(\beta_3) + \cdots + s_{\beta_1}s_{\beta_2}\dots s_{\beta_{m-1}}(\beta_m).$$
Moreover, this is the unique way in which $-w\cdot 0$ can be expressed as a sum of distinct positive roots.

2.3. The Induction Functor and Filtrations. Let $\mathcal{G}_r(k) := \operatorname{ind}_{G(\mathbb{F}_q)}^G(k)$. The functor $\mathcal{G}_r(-)$ is exact and one can use Frobenius reciprocity to relate extensions over G with extensions over $G(\mathbb{F}_q)$ [BNP, Proposition 2.2].

PROPOSITION 2.3.1. *Let M, N be rational G-modules. Then, for all $i \geq 0$,*
$$\operatorname{Ext}^i_{G(\mathbb{F}_q)}(M,N) \cong \operatorname{Ext}^i_G(M, N\otimes \mathcal{G}_r(k)).$$

In order to make the desired computations of cohomology groups, we will make use of Proposition 2.3.1 (with $M = k = N$). In addition, we will use a special filtration on $\mathcal{G}_r(k)$ (cf. [BNP, Proposition 2.4.1]).

PROPOSITION 2.3.2. *The induced module $\mathcal{G}_r(k)$ has a filtration with factors of the form $H^0(\lambda)\otimes H^0(\lambda^*)^{(r)}$ with multiplicity one for each $\lambda \in X(T)_+$.*

2.4. A Vanishing Criterion. The filtration from Proposition 2.3.2 allows one to obtain a condition on G-cohomology which leads to vanishing of $G(\mathbb{F}_{p^r})$-cohomology (cf. [BNP, Corollary 2.6.1]).

PROPOSITION 2.4.1. *Let m be the least positive integer such that there exists $\lambda \in X(T)_+$ with $\mathrm{H}^m(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(r)}) \neq 0$. Then $\mathrm{H}^i(G(\mathbb{F}_q), k) \cong \mathrm{H}^i(G, \mathcal{G}_r(k)) = 0$ for $0 < i < m$.*

2.5. Non-vanishing. While the identification of an m satisfying Proposition 2.4.1 gives a vanishing range as in (1.1.1), it does not a priori follow that $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$. The following theorem provides conditions which assist with addressing (1.1.2) or (1.1.3) [BNP, Theorem 2.8.1].

THEOREM 2.5.1. *Let m be the least positive integer such that there exists $\nu \in X(T)_+$ with $\mathrm{H}^m(G, H^0(\nu)\otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in X(T)_+$ be such that $\mathrm{H}^m(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $\mathrm{H}^{m+1}(G, H^0(\nu)\otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ that are linked to λ. Then*
(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < m$;
(b) $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$;
(c) *if, in addition,* $\mathrm{H}^m(G, H^0(\nu)\otimes H^0(\nu^*)^{(r)}) = 0$ *for all* $\nu \in X(T)_+$ *with* $\nu \neq \lambda$, *then*
$$\mathrm{H}^m(G(\mathbb{F}_q), k) \cong \mathrm{H}^m(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(r)}).$$

From the filtration on $\mathcal{G}_r(k)$ in Proposition 2.3.2, $\mathrm{H}^i(G(\mathbb{F}_q), k) \cong \mathrm{H}^i(G, \mathcal{G}_r(k))$ can be decomposed as a direct sum over linkage classes of dominant weights. For a fixed linkage class $\mathcal{L}$, let m be the least positive integer such that there exists $\nu \in \mathcal{L}$ with $\mathrm{H}^m(G, H^0(\nu)\otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in \mathcal{L}$ be such that $\mathrm{H}^m(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $\mathrm{H}^{m+1}(G, H^0(\nu)\otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ in $\mathcal{L}$. Then analogous to Theorem 2.5.1, it follows that $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$. See [BNP, Theorem 2.8.2].

2.6. Reducing to G_1-cohomology. From Sections 2.4 and 2.5, the key to understanding the vanishing of $\mathrm{H}^i(G(\mathbb{F}_{p^r}), k)$ is to understand $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(r)})$ for all dominant weights λ. For $r = 1$, these groups can be related to G_1-cohomology groups (cf. [BNP, Lemma 3.1]).

LEMMA 2.6.1. *Suppose $p > h$ and let $\nu_1, \nu_2 \in X(T)_+$. Then for all j*
$$\mathrm{H}^j(G, H^0(\nu_1)\otimes H^0(\nu_2^*)^{(1)}) \cong \operatorname{Ext}^j_G(V(\nu_2)^{(1)}, H^0(\nu_1)) \cong \operatorname{Hom}_G(V(\nu_2), \mathrm{H}^j(G_1, H^0(\nu_1))^{(-1)}).$$

We remark that the aforementioned lemma would hold for arbitrary rth-twists if it was known that the cohomology group $\mathrm{H}^j(G_r, H^0(\nu))^{(-r)}$ admits a good filtration, which is a long-standing conjecture of Donkin. For $p > h$, this is known for $r = 1$ by results of Andersen-Jantzen [AJ] and Kumar-Lauritzen-Thomsen [KLT]. In that case, the lemma is only needed when $\nu_1 = \nu_2$. For arbitrary r we can often work inductively from the $r = 1$ case. This requires slightly more general Ext-computations and the possibility that $\nu_1 \neq \nu_2$.

2.7. Dimensions for $r = 1$. From Lemma 2.6.1, for $\nu \in X(T)_+$, the cohomology group $\mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(1)})$ can be identified with $\mathrm{Hom}_G(V(\nu), \mathrm{H}^i(G_1, H^0(\nu)^{(-1)})$. It is well-known that, from block considerations, $\mathrm{H}^i(G_1, H^0(\nu)) = 0$ unless $\nu = w \cdot 0 + p\mu$ for $w \in W$ and $\mu \in X(T)$. For $p > h$, from [AJ] and [KLT], we have

$$\mathrm{H}^i(G_1, H^0(\nu))^{(-1)} = \begin{cases} \mathrm{ind}_B^G(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu) & \text{if } \nu = w \cdot 0 + p\mu \\ 0 & \text{otherwise,} \end{cases} \tag{2.7.1}$$

where $\mathfrak{u} = \mathrm{Lie}(U)$. Note also that, since $p > h$ and ν is dominant, μ must also be dominant.

For a dominant weight $\nu = p\mu + w \cdot 0$, observe that, from Lemma 2.6.1 and (2.7.1), we have

$$\begin{aligned} \mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(1)}) &\cong \mathrm{Hom}_G(V(\nu), \mathrm{H}^i(G_1, H^0(\nu))^{(-1)}) \\ &\cong \mathrm{Hom}_G(V(\nu), \mathrm{ind}_B^G(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu)) \\ &\cong \mathrm{Hom}_B(V(\nu), S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu). \end{aligned}$$

Hence, if $\mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(1)}) \neq 0$, then $\nu - \mu = (p-1)\mu + w \cdot 0$ must be a sum of $(i - \ell(w))/2$ positive roots.

For a weight ν and $n \geq 0$, let $P_n(\nu)$ denote the dimension of the ν-weight space of $S^n(\mathfrak{u}^*)$. Equivalently, for $n > 0$, $P_n(\nu)$ denotes the number of times that ν can be expressed as a sum of exactly n positive roots, while $P_0(0) = 1$. The function P_n is often referred to as *Kostant's Partition Function*. By using [AJ, 3.8], [KLT, Thm 2], Lemma 2.6.1, and (2.7.1), we can give an explicit formula for the dimension of $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$ (cf. [BNP, Proposition 3.2.1, Corollary 3.5.1]).

Proposition 2.7.1. *Let $p > h$ and $\lambda = p\mu + w \cdot 0 \in X(T)_+$. Then*

$$\dim \mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot \lambda - \mu).$$

2.8. Degree Bounds. The following gives a fundamental constraint on non-zero i such that

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \mathrm{Ext}_G^i(V(\lambda)^{(1)}, H^0(\lambda)) \neq 0.$$

It is stated in a more general Ext-context as it will also be used in some inductive arguments for $r > 1$ (cf. [BNP, Proposition 3.4.1]).

Proposition 2.8.1. *Let $p > h$ with $\gamma_1, \gamma_2 \in X(T)_+$, both non-zero, such that $\gamma_j = p\delta_j + w_j \cdot 0$ with $\delta_j \in X(T)_+$ and $w_j \in W$ for $j = 1, 2$. Assume $\mathrm{Ext}_G^i(V(\gamma_2)^{(1)}, H^0(\gamma_1)) \neq 0$.*

(a) *Let $\sigma \in \Phi^+$. If Φ is of type G_2, assume that σ is a long root. Then $p\langle \delta_2, \sigma^\vee \rangle - \langle \delta_1, \sigma^\vee \rangle + \ell(w_1) + \langle w_2 \cdot 0, \sigma^\vee \rangle \leq i$.*

(b) *If $\tilde{\alpha}$ denotes the longest root in Φ^+, then $p\langle \delta_2, \tilde{\alpha}^\vee \rangle - \langle \delta_1, \tilde{\alpha}^\vee \rangle + \ell(w_1) - \ell(w_2) - 1 \leq i$. Equality requires that $\gamma_2 - \delta_1 = ((i - \ell(w_1))/2)\tilde{\alpha}$ and $\langle -w_2 \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(w_2) + 1$.*

(c) *If $\gamma_1 = \gamma_2 = p\delta + w \cdot 0$, then $i \geq (p-1)\langle \delta, \tilde{\alpha}^\vee \rangle - 1$.*

Proposition 2.8.1 can be generalized to the following (cf. [BNP, Proposition 4.3.1]).

Proposition 2.8.2. *Let $p > h$, $0 \neq \lambda \in X(T)_+$ and $i \geq 0$. If $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$, then there exists a sequence of non-zero weights $\lambda = \gamma_0, \gamma_1, \dots, \gamma_{r-1}, \gamma_r = \lambda \in X(T)_+$ such that $\gamma_j = p\delta_j + u_j \cdot 0$ for some $u_j \in W$ and nonzero $\delta_j \in X(T)_+$. Moreover, for each $1 \leq j \leq r$, there exists a nonnegative integer l_j with $\mathrm{Ext}_G^{l_j}(V(\gamma_j)^{(1)}, H^0(\gamma_{j-1})) \neq 0$ and $\sum_{j=1}^r l_j = i$. Furthermore,*

$$i \geq \left(\sum_{j=1}^{r} (p-1)\langle \delta_j, \tilde{\alpha}^\vee \rangle \right) - r. \tag{2.8.1}$$

Equality requires that $p\delta_j - \delta_{j-1} + u_j \cdot 0 = ((l_j - \ell(u_{j-1}))/2)\tilde{\alpha}$ and that $\langle -u_j \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(u_j) + 1$ for all $1 \leq j \leq r$.

Note that the assumption $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$ in the proposition can be replaced by

$$\mathrm{Ext}_{G/G_1}^k(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\lambda))) \neq 0,$$

where $k + l = i$. In that case one arrives at the same conclusions with $l_1 = l$.

3. Vanishing Ranges in the Simply Laced Case

In this section we obtain some general vanishing information for those cases when the root system Φ is simply laced. In such cases, the longest root $\tilde{\alpha}$ and the longest short root α_0 coincide. Following the discussion in Section 2, we want to consider when

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$$

for $i > 0$ and $\lambda \in X(T)_+$.

3.1. To gain information on such cohomology groups, we will use Lemma 2.6.1 and (2.7.1). The following proposition will aid us in showing that certain cohomology groups are non-zero.

PROPOSITION 3.1.1. *Let $\tilde{\alpha}$ denote the longest root of Φ and l be a nonnegative integer. Then*

$$\mathrm{Hom}_G(V((l+1)\tilde{\alpha}), \mathrm{ind}_B^G(S^l(\mathfrak{u}^*) \otimes \tilde{\alpha})) \cong k.$$

PROOF. The claim follows from the diagram below and the fact that all modules in the commutative diagram below have a one-dimensional highest weight space with weight $(l+1)\tilde{\alpha}$. The first embedding is a consequence of the fact that the module $\underbrace{V(\tilde{\alpha}) \otimes \cdots \otimes V(\tilde{\alpha})}_{(l+1)\text{ times}}$ has a Weyl module filtration. The Weyl module $V(\tilde{\alpha})$ is isomorphic to the dual of the adjoint representation, $\mathfrak{g}^*$. Clearly, $\mathfrak{g}^*$ maps onto $\mathfrak{u}^*$ and $V(\tilde{\alpha})$ maps onto $\phi_{-\tilde{\alpha}}$ (of weight $\tilde{\alpha}$) as B-modules. Hence, we obtain the two B-surjections in the first line of the diagram. The remaining maps and the commutativity of the diagram arise via the universal property of the induction functor.

$$\begin{tikzcd} V((l+1)\tilde{\alpha}) \arrow[r, hook] & \underbrace{V(\tilde{\alpha}) \otimes \cdots \otimes V(\tilde{\alpha})}_{(l+1)\text{ times}} \arrow[r, two heads] \arrow[drr] & \underbrace{\mathfrak{u}^* \otimes \cdots \otimes \mathfrak{u}^*}_{l\text{ times}} \otimes \tilde{\alpha} \arrow[r, two heads] & S^l(\mathfrak{u}^*) \otimes \tilde{\alpha} \\ & & & \mathrm{ind}_B^G(S^l(\mathfrak{u}^*) \otimes \tilde{\alpha}) \arrow[u] \end{tikzcd}$$

□

3.2. For a G-module V and a dominant weight γ let $[V]_\gamma$ denote the unique maximal summand of V whose composition factors have highest weights linked to γ.

LEMMA 3.2.1. *Assume that the root system Φ of G is simply laced. Let $\tilde{\alpha}$ denote the longest root and define $\lambda = p\tilde{\alpha} + s_{\tilde{\alpha}} \cdot 0 = (p-h+1)\tilde{\alpha}$. Then*

(a) *for any non-zero dominant weight μ linked to zero we have* $\mathrm{H}^i(G, H^0(\mu) \otimes H^0(\mu^*)^{(r)}) = 0$ *whenever $i < r(2p-3)$;*
(b) *for any non-zero dominant weight μ linked to zero we have* $\mathrm{Ext}^k_{G/G_1}(V(\mu)^{(r)}, \mathrm{H}^l(G_1, H^0(\mu))) = 0$ *whenever $k + l < r(2p-3)$;*
(c) $[\mathrm{H}^i(G_1, H^0(\lambda))^{(-1)}]_0 \cong \begin{cases} H^0(\lambda) & \text{if } i = 2p-3 \\ 0 & \text{if } 0 < i < 2p-3. \end{cases}$

PROOF. We apply Proposition 2.8.2. Note that μ being linked to zero forces all the weights δ_j of Proposition 2.8.2 to be in the root lattice. This forces $\langle \delta_j, \tilde{\alpha}^\vee \rangle \geq 2$. Parts (a) and (b) now follow from equation (2.8.1) and the remark in Section 2.8.

For part (c) we make use of Proposition 3.1.1 with $l + 1 = p - h + 1$ and conclude that

$$\mathrm{Hom}_G(V(\lambda), \mathrm{ind}_B^G(S^{(p-h)}(\mathfrak{u}^*) \otimes \tilde{\alpha})) \cong k.$$

Note that in the simply laced case $\ell(s_{\tilde{\alpha}}) = 2h - 3$ which combined with (2.7.1) yields $\mathrm{ind}_B^G(S^{p-h}(\mathfrak{u}^*) \otimes \tilde{\alpha})) \cong \mathrm{H}^{2p-3}(G_1, H^0(\lambda)^{(-1)})$.

The weight λ is the smallest non-zero dominant weight in the zero linkage class. Any other non-zero weight μ in the linkage class will be of the form $\mu = \lambda + \sigma = (p-h+1)\tilde{\alpha} + \sigma$, where σ is a non-zero sum of positive roots. Clearly μ cannot be a weight of $S^m(\mathfrak{u}^*) \otimes \tilde{\alpha}$ whenever $m \leq p - h$. Hence,

$$\mathrm{Hom}_G(V(\mu), \mathrm{H}^i(G_1, H^0(\lambda))^{(-1)}) \cong \mathrm{Hom}_G(V(\mu), \mathrm{ind}_B^G(S^{\frac{i-\ell(s_{\tilde{\alpha}})}{2}}(\mathfrak{u}^*) \otimes \tilde{\alpha})) = 0$$

for all $0 < i \le 2p-3$. Since $\mathrm{H}^{2p-3}(G_1, H^0(\lambda))^{(-1)}$ has a good filtration one obtains

$$[\mathrm{H}^{2p-3}(G_1, H^0(\lambda))^{(-1)}]_0 \cong H^0(\lambda).$$

Part (b) now implies that $[\mathrm{H}^i(G_1, H^0(\lambda))]_0 = 0$ whenever $0 < i < 2p-3$. □

THEOREM 3.2.2. *Assume that the root system of G is simply laced. Then*

$$\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0.$$

PROOF. Let μ be a weight in the zero linkage class. From Lemma 3.2.1(a), we know that $\mathrm{H}^i(G, H^0(\mu) \otimes H^0(\mu^*)^{(1)}) = 0$ for $i < r(2p-3)$. Let $\lambda = (p-h+1)\tilde{\alpha}$. We next show by induction on r that $\mathrm{H}^{r(2p-3)}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. For $r = 1$, this follows from Lemma 2.6.1 and Lemma 3.2.1(c).

For $r > 1$, we look at the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{k,l} = \mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\lambda))) \Rightarrow \mathrm{Ext}^{k+l}_G(V(\lambda)^{(r)}, H^0(\lambda)).$$

Lemma 3.2.1(b) implies that the $E_2^{k,l} = 0$ for $k + l < r(2p-3)$. Note that

$$E_2^{k,l} = \mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, \mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}).$$

Lemma 3.2.1(c) implies that $E_2^{k,l} = 0$ for $l < 2p-3$, and, moreover, that $H^0(\lambda)$ is a summand of $\mathrm{H}^{2p-3}(G_1, H^0(\lambda))^{(-1)}$. Hence, (cf. [BNP, Lemma 5.4]),

$$E_2^{(r-1)(2p-3),2p-3} = \mathrm{Ext}^{(r-1)(2p-3)}_G(V(\lambda)^{(r-1)}, \mathrm{H}^{2p-3}(G_1, H^0(\lambda))^{(-1)})$$

has a summand isomorphic to $\mathrm{Ext}^{(r-1)(2p-3)}_G(V(\lambda)^{(r-1)}, H^0(\lambda))$. By induction, this Ext-group is non-zero, and hence $E_2^{(r-1)(2p-3),2p-3} \neq 0$ and transgresses to the E_∞-page, which implies that

$$\mathrm{Ext}^{r(2p-3)}_G(V(\lambda)^{(r)}, H^0(\lambda)) \neq 0.$$

Since λ is the lowest non-zero dominant weight in the zero linkage class, the claim now follows by applying the argument given in Section 2.5 to the weight λ and the zero linkage class. □

3.3. Finite groups of adjoint type. In this section we assume that G is simply laced and of adjoint type. The fixed points of the rth iterated Frobenius map on G will again be denoted by $G(\mathbb{F}_q)$. For example, if G is the adjoint group of type A then $G(\mathbb{F}_q)$ is the projective linear group with entries in the field with q elements. Propositions 2.3.1 and 2.3.2 can also be applied to groups of adjoint type. Note that the root lattice and the weight lattice coincide in this case. Therefore all dominant weights of the form $p\delta + w \cdot 0$ are automatically in the zero linkage class. From Lemma 3.2.1, Theorem 3.2.2, and Proposition 2.4.1, one obtains the following corollary.

COROLLARY 3.3.1. *Assume that the root system Φ of G is simply laced and that G is of adjoint type. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$;
(b) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.

Note that, for type A_n and $q-1$ and $n+1$ being relatively prime, the adjoint and the universal types of the finite groups coincide. In these cases the above claim was already observed in Theorems 6.5.1 and 6.14.1 of [BNP].

REMARK 3.3.2. Let G be of type A, D, E_6 and of adjoint type. Let σ denote an automorphism of the Dynkin diagram of G. Then σ induces a group automorphism of G that commutes with the Frobenius morphism, which we will also denote by σ. Let $G_\sigma(\mathbb{F}_q)$ be the finite group consisting of the fixed points of σ composed with F. Note that σ fixes the maximal root $\tilde{\alpha}$. Therefore the discussion in this section also applies to the twisted groups $G_\sigma(\mathbb{F}_q)$ of adjoint type. In particular, Corollary 3.3.1 holds for these groups as well.

4. Type D_n, $n \geq 4$

Assume throughout this section that Φ is of type D_n, $n \geq 4$, and that $p > h = 2n-2$. Following Section 2, our goal is to find the least $i > 0$ such that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)) \neq 0$ for some λ.

4.1. Restrictions. Suppose that $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i>0$ and $\lambda = p\mu + w\cdot 0$ with $\mu \in X(T)_+$ and $w\in W$. From Proposition 2.8.1(c), $i \geq (p-1)\langle \mu, \tilde{\alpha}^\vee\rangle - 1$.

For a fundamental dominant weight ω_j,

$$\langle \omega_j, \tilde{\alpha}^\vee\rangle = \begin{cases} 1 & \text{if } j = 1, n-1, n \\ 2 & \text{if } 2 \leq j \leq n-2. \end{cases}$$

Therefore, if $\mu \neq \omega_1, \omega_{n-1}, \omega_n$, we will have $\langle \mu, \tilde{\alpha}^\vee\rangle \geq 2$ and $i \geq 2p-3$. This reduces us to analyzing the cases when $\mu = \omega_1, \omega_{n-1}, \omega_n$,

4.2. The case of ω_1. We consider first the case that $\lambda = p\omega_1 + w\cdot 0$ and obtain the following restrictions.

LEMMA 4.2.1. *Suppose Φ is of type D_n with $n\geq 4$ and $p > 2n-2$. Suppose $\lambda = p\omega_1 + w\cdot 0 \in X(T)_+$ with $w \in W$. Then*

(a) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-2n$*;*
(b) *if* $\mathrm{H}^{2p-2n}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$*, then* $\lambda = p\omega_1 - (2n-2)\omega_1 = (p-2n+2)\omega_1$.

PROOF. Following the discussion in Section 2.7, $\lambda - \omega_1 = (p-1)\omega_1 + w\cdot 0$ must be a weight of $S^j(\mathfrak{u}^*)$ for $j = \frac{i-\ell(w)}{2}$. Recall that $\omega_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n$. Consider the decomposition of $-w\cdot 0$ into a sum of $\ell(w)$ distinct positive roots (see Observation 2.2.1). Write $\ell(w) = a+b$ where a is the number of positive roots in this decomposition that contain α_1 and b is the number of roots in this decomposition that do not contain α_1. Then $\lambda - \omega_1$ contains $p-1-a$ copies of α_1. Since any root contains at most one copy of α_1, we have

$$\frac{i-\ell(w)}{2} = j \geq p-1-a.$$

Replacing $\ell(w)$ by $a+b$ and simplifying gives

$$i \geq 2p-2-a+b.$$

The total number of positive roots containing an α_1 is $2n-2$. Since we necessarily then have $a \leq 2n-2$, we can rewrite the above as

$$\begin{aligned} i &\geq 2p-2-(2n-2)+b \\ &= 2p-2n+b \\ &\geq 2p-2n \end{aligned}$$

since $b\geq 0$. This proves part (a). Furthermore, we see that $i = 2p-2n$ if and only if $b = 0$ and $a = 2n-2$. In other words, when $-w\cdot 0$ is expressed as a sum of distinct positive roots, it consists precisely of all $2n-2$ roots which contain an α_1. That is, $-w\cdot 0 = (2n-2)\omega_1$, which gives part (b). □

4.3. The case of ω_1 continued. We will show in Proposition 4.3.2 that

$$\mathrm{H}^{2p-2n}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$$

for $\lambda = (p-2n+2)\omega_1$. To do this, we will make use of Proposition 2.7.1. We first make some observations about relevant partition functions. Note that $\omega_1 = \epsilon_1$.

For Φ of type D_n, with $n\geq 4$, and integers m, k, we set

$$P(m,k,n) := \begin{cases} \sum_{u\in W}(-1)^{\ell(u)} P_k(u\cdot m\epsilon_1) & \text{if } m\geq 1, k\geq 0, \\ 1 & \text{if } m = 0, k = 0, \\ 0 & \text{else.} \end{cases}$$

Note that

$$P(m,k,n) = \dim \mathrm{Hom}_G(V(m\epsilon_1), H^0(G/B, S^k(\mathfrak{u}^*))) = [\mathrm{ch}\, H^0(G/B, S^k(\mathfrak{u}^*)) : \mathrm{ch}\, H^0(m\epsilon_1)],$$

when $m\geq 0, k\geq 0, n\geq 4$.

LEMMA 4.3.1. *Suppose Φ is of type D_n with $n\geq 4$ and $m\geq 0$.*

(a) $P(m,k,n) = 0$ *whenever* $k < m$.
(b) $\sum_{u\in W}(-1)^{\ell(u)} P_k(u\cdot m\epsilon_1 - \epsilon_1) = P(m-1,k,n)$.
(c) $\sum_{u\in W}(-1)^{\ell(u)} P_k(u\cdot m\epsilon_1 + \epsilon_1) = P(m+1,k,n) - P(m+1,k,n-1)$, *for* $n\geq 5$.

(d) $\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1)=P(m-1,k-2n+2,n)$.
(e) $P(m,k,n)=P(m,k,n-1)+P(m-2,k-2n+2,n)$, *for* $m\geq 0, n\geq 5$.
(f) $P(m,m,n)=1$, *for* $n\geq 4$ *and* m *even.*

PROOF. (a) The weight $\omega_1=\epsilon_1=\frac{1}{2}(\alpha_1+\tilde{\alpha})$, written as a sum of simple roots, contains one copy of α_1. Assume that $0\leq k<m$ and $n\geq 4$. Recall that $P(m,k,n)=\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1)$. Note that $u\cdot m\epsilon_1=u(m\epsilon_1)+u\cdot 0$ is a sum of positive roots if and only if $u(\epsilon_1)=\epsilon_1$. Therefore, $P(m,k,n)=\sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1)$. If $u(\epsilon_1)=\epsilon_1$ then $u\cdot m\epsilon_1=m\epsilon_1+u\cdot 0$. In addition $-u\cdot 0$, written as a sum of simple roots, contains no α_1. Hence, $u\cdot m\epsilon_1$, written as a sum of simple roots, contains exactly m copies of α_1. Each positive root of Φ contains at most one copy of α_1. Therefore at least m positive roots are needed to sum up to $u\cdot m\epsilon_1$. One concludes that $P_k(u\cdot m\epsilon_1)=0$ for $k<m$.

(b) Again $u\cdot m\epsilon_1-\epsilon_1$ is a sum of positive roots only if $u(\epsilon_1)=\epsilon_1$. Therefore,

$$\begin{aligned}\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1-\epsilon_1) &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1-\epsilon_1)\\ &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k((m-1)\epsilon_1+u\cdot 0)\\ &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot (m-1)\epsilon_1)\\ &= P(m-1,k,n).\end{aligned}$$

(c) For the expression $u\cdot m\epsilon_1+\epsilon_1$ to be a sum of positive roots one needs either $u(\epsilon_1)=\epsilon_1$ or $u(\epsilon_1)=\epsilon_2$ and $u(\epsilon_2)=\epsilon_1$. Set $A=\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1)$ Then

$$\begin{aligned}A &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1)+\sum_{\{u\in W|u(\epsilon_1)=\epsilon_2,u(\epsilon_2)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1)\\ &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k((m+1)\epsilon_1+u\cdot 0)\\ &\qquad +\sum_{\{u\in W|u(\epsilon_1)=\epsilon_1,u(\epsilon_2)=\epsilon_2\}}(-1)^{\ell(u)+1}P_k(s_{\alpha_1}m\epsilon_1+u\cdot 0-\alpha_1+\epsilon_1)\\ &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot (m+1)\epsilon_1)-\sum_{\{u\in W|u(\epsilon_1)=\epsilon_1,u(\epsilon_2)=\epsilon_2\}}(-1)^{\ell(u)}P_k(u\cdot (m+1)\epsilon_2)\\ &= P(m+1,k,n)-P(m+1,k,n-1).\end{aligned}$$

(d) We make use of the fact that $\omega_1=\epsilon_1$ is a minuscule weight and obtain:

$$\begin{aligned}A &= [(\sum_{i\geq 0}(-1)^i\operatorname{ch}H^i(G/B,S^k(\mathfrak{u}^*)\otimes -\epsilon_1)):\operatorname{ch}H^0(m\epsilon_1)]\\ &= [\operatorname{ch}H^0(G/B,S^k(\mathfrak{u}^*)\otimes -\epsilon_1)):\operatorname{ch}H^0(m\epsilon_1)]\\ &= [\operatorname{ch}H^0(G/B,S^{k-2n+2}(\mathfrak{u}^*)\otimes \epsilon_1)):\operatorname{ch}H^0(m\epsilon_1)] \qquad \text{(by [KLT, Lemma 6])}\\ &= \sum_{u\in W}(-1)^{\ell(u)}P_{k-2n-2}(u\cdot m\epsilon_1-\epsilon_1) \qquad \text{(by [AJ, 3.8])}\\ &= P(m-1,k-2n+2,n) \qquad \text{(by (b))}.\end{aligned}$$

Part (e) now follows directly from (c) and (d).

(f) If $n\geq 5$, it follows from (e) and (a) that $P(m,m,n)=P(m,m,n-1)$. So the claim holds if it holds for

$n = 4$. If $n = 4$ and $k = m$, then (e) has to be replaced by

$$P(m,m,4) = \sum_{\{u\in W \mid u(\epsilon_1)=\epsilon_1, u(\epsilon_2)=\epsilon_2\}} (-1)^{\ell(u)} P_m(u\cdot m\epsilon_2).$$

Note that the both sides of the equation are zero unless m is even. For even m, a direct computation similar to that in the proof of [BNP, Lemma 6.11] shows that

$$\sum_{\{u\in W \mid u(\epsilon_1)=\epsilon_1, u(\epsilon_2)=\epsilon_2\}} (-1)^{\ell(u)} P_m(u\cdot m\epsilon_2) = 1.$$

□

PROPOSITION 4.3.2. *Suppose Φ is of type D_n with $n \geq 4$. Assume that $p > 2n-2$. Let $\lambda = (p-2n+2)\omega_1$. Then*

$$\mathrm{H}^{2p-2n}(G, H^0(\lambda)\otimes H^0(\lambda)^{(1)}) = k.$$

PROOF. From the previous discussion we know that $\lambda = (p-2n+2)\omega_1$ is of the form $p\omega_1 + w\cdot 0$ with $\ell(w) = 2n-2$. Set $k = (i - l(w))/2$. From Proposition 2.7.1 and Lemma 4.3.1(b), one concludes

$$\dim \mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda)^{(1)}) = [\mathrm{ch}\, H^0(G/B, S^k(\mathfrak{u}^*)\otimes\omega_1) : \mathrm{ch}\, H^0(\lambda)] = P(p-2n+1, k, n).$$

By Lemma 4.3.1(a), this expression is zero unless $k \geq p-2n+1$ and Lemma 4.3.1(d) it follows that $P(p-2n+1, p-2n+1, n) = 1$. Replacing k by $(i-2n+2)/2$ and solving for i yields the claim. □

4.4. The case of ω_{n-1} and ω_n. We now consider the case that $\lambda = p\omega_{n-1} + w\cdot 0$ or $\lambda = p\omega_n + w\cdot 0$ for $w \in W$ with $\lambda \in X(T)_+$.

LEMMA 4.4.1. *Suppose Φ is of type D_n with $n \geq 4$ and $p > 2n-2$. Suppose $\lambda = p\omega_{n-1} + w\cdot 0 \in X(T)_+$ or $\lambda = p\omega_n + w\cdot 0 \in X(T)_+$ with $w \in W$, and $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $i \neq 0$. Then*

(a) *$i \geq \dfrac{(p-n)n}{2}$;*
(b) *if $n \geq 5$, then $i \geq 2p-2n+2$.*

PROOF. We consider the case of ω_n. By symmetry, the case of ω_{n-1} can be dealt with in a similar manner. Following the discussion in Section 2.7, $\lambda - \omega_n = (p-1)\omega_n + w\cdot 0$ must be a weight of $S^j(\mathfrak{u}^*)$ for $j = \frac{i-\ell(w)}{2}$. Recall that

$$\omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + (n-2)\alpha_{n-2}) + \frac{(n-2)}{4}\alpha_{n-1} + \frac{n}{4}\alpha_n.$$

Consider the decomposition of $-w\cdot 0$ into a sum of distinct positive roots (cf. Observation 2.2.1). Write $\ell(w) = a+b$ where a is the number of positive roots in this decomposition which contain α_n and b is the number of roots in this decomposition which do not contain α_n. Then $\lambda - \omega_n$ contains $\frac{(p-1)n}{4} - a$ copies of α_n. Since any root contains at most one copy of α_n, we have

$$\frac{i-\ell(w)}{2} = j \geq \frac{(p-1)n}{4} - a.$$

Substituting $\ell(w) = a+b$, rewriting, and simplifying, we get

$$i \geq \frac{(p-1)n}{2} - a + b.$$

The total number of positive roots containing α_n is $\frac{(n-1)n}{2}$. Since we necessarily have $a \leq \frac{(n-1)n}{2}$ and $b \geq 0$, we get

$$\begin{aligned} i &\geq \frac{(p-1)n}{2} - \frac{(n-1)n}{2} + b \\ &= \frac{(p-n)n}{2} + b \\ &\geq \frac{(p-n)n}{2} \end{aligned}$$

which gives part (a).

For part (b), assume that $n \geq 5$. We want to show that

$$\frac{(p-n)n}{2} \geq 2p - 2n + 2.$$

This is equivalent to showing that $(p-n)n \geq 4p - 4n + 4$. Consider the left hand side:

$$(p-n)n = np - n^2 = 4p + (n-4)p - n^2.$$

Hence the problem is reduced to showing that $(n-4)p - n^2 \geq -4n + 4$ or $(n-4)p - n^2 + 4n - 4 \geq 0$. Since $p \geq 2n-1$, we have

$$\begin{aligned}(n-4)p - n^2 + 4n - 4 &\geq (n-4)(2n-1) - n^2 + 4n - 4 \\ &= n^2 - 5n = n(n-5) \geq 0\end{aligned}$$

since $n \geq 5$. Part (b) follows. $\square$

Note that if $n = 4$,

$$\frac{(p-n)n}{2} = \frac{4(p-4)}{2} = 2p - 8 = 2p - 2n.$$

4.5. Summary for type D. The following two theorems summarize our findings when the root system is of type D_n.

THEOREM 4.5.1. *Suppose Φ is of type D_n with $n \geq 4$. Assume that $p > 2n-2$. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 2n$;

(b) $\mathrm{H}^{2p-2n}(G(\mathbb{F}_p), k) = \begin{cases} k \ \text{if}\ n \geq 5 \\ k \oplus k \oplus k \ \text{if}\ n = 4. \end{cases}$

PROOF. Part (a) follows from Section 4.1, Lemma 4.2.1(a), Lemma 4.4.1, and Proposition 2.4.1.

For part (b), when $n \geq 5$, it follows from Section 4.1, Lemma 4.2.1, Proposition 4.3.2 and Lemma 4.4.1 that $\lambda = (2p-2n+2)\omega_1$ is the only dominant weight with $\mathrm{H}^{2p-2n}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Since λ is the lowest weight in its linkage class, the claim follows from Theorem 2.5.1. For $n = 4$ the symmetry of the root system yields $\mathrm{H}^{2p-2n}(G, H^0(\lambda) \otimes H^0(\lambda)^{(1)}) = k$ for the weights $\lambda = (p-6)\omega_1$, $\lambda = (p-6)\omega_3$ and $\lambda = (p-6)\omega_4$, and those are the only weights with non-zero G-cohomology in degree $2p-2n$. Each weight is minimal in its own linkage class. The claim follows. $\square$

Working inductively from the $r = 1$ case, we can obtain sharp vanishing bounds for arbitrary r.

THEOREM 4.5.2. *Suppose Φ is of type D_n with $n \geq 4$. Assume that $p > 2n-2$. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p - 2n)$;

(b) $\mathrm{H}^{r(2p-2n)}(G(\mathbb{F}_q), k) = \begin{cases} k \ \text{if}\ n \geq 5 \\ k \oplus k \oplus k \ \text{if}\ n = 4. \end{cases}$

PROOF. For part (a), we need to show that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ for $0 < i < r(2p-2n)$ and $\lambda \in X(T)_+$. If that is true, then the claim follows from Proposition 2.4.1. For part (b), we require precise information on those dominant weights λ for which $\mathrm{H}^{r(2p-2n)}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. The root lattice $\mathbb{Z}\Phi$ has four cosets within $X(T)$: $\mathbb{Z}\Phi$, $\{\omega_1 + \mathbb{Z}\Phi\}$, $\{\omega_{n-1} + \mathbb{Z}\Phi\}$, and $\{\omega_n + \mathbb{Z}\Phi\}$. If λ is a weight in the root lattice claim (a) follows from Lemma 3.2.1(a). Furthermore, no such weights can contribute to cohomology in degree $r(2p-2n)$.

Assume that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$ for some $i > 0$ and apply Proposition 2.8.2. Suppose first that $\lambda = p\delta_0 + u_0 \cdot 0$ with $\delta_0 = \delta_r \in \{\omega_1 + \mathbb{Z}\Phi\}$. For each $1 \leq j \leq r$, in order to have $\mathrm{Ext}_G^{l_j}(V(\gamma_j)^{(1)}, H^0(\gamma_{j-1})) \neq 0$, then $\gamma_j - \delta_{j-1} = p\delta_j + u_j \cdot 0 - \delta_{j-1}$ must be a weight in $S^{\frac{l_j - \ell(u_{j-1})}{2}}(\mathfrak{u}^*)$. This implies that $p\delta_j - \delta_{j-1}$ must lie in the positive root lattice. Since $\delta_r \in \{\omega_1 + \mathbb{Z}\Phi\}$, we necessarily have $p\delta_r \in \{\omega_1 + \mathbb{Z}\Phi\}$. Since $p\delta_r - \delta_{r-1} \in \mathbb{Z}\Phi$, it then follows that $\delta_{r-1} \in \{\omega_1 + \mathbb{Z}\Phi\}$. Inductively one concludes that $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\}$ for all j.

For a weight γ, when expressed as a sum of simple roots, let $N(\gamma)$ denote the number of copies of α_1 that appear. Since a positive root contains at most one copy of α_1, we have

$$\frac{l_j - \ell(u_{j-1})}{2} \geq pN(\delta_j) - N(-u_j \cdot 0) - N(\delta_{j-1}).$$

From Observation 2.2.1, we know that $-u_j\cdot 0$ can be expressed uniquely as $\ell(u_j)$ distinct positive roots. Write $\ell(u_j)=a_j+b_j$ where a_j denotes the number of roots containing an α_1 and b_j the number that do not. Then $N(u_j\cdot 0)=-a_j$, and rewriting the above gives

$$l_j \geq 2pN(\delta_j)-2N(\delta_{j-1})-2a_j+a_{j-1}+b_{j-1}.$$

Hence, summing over j gives

$$i=\sum_{j=1}^{r} l_j \geq \sum_{j=1}^{r}(2p-2)N(\delta_j)-\sum_{j=1}^{r} a_j+\sum_{j=1}^{r} b_j.$$

Recall that the δ_j are non-zero dominant weights. By the assumption on δ_j, $N(\delta_j)\geq 1$. The total number of positive roots containing an α_1 is $2n-2$. Hence, $a_j\leq 2n-2$. With this, we get

$$i\geq r(2p-2)-r(2n-2)+\sum_{j=1}^{r} b_j = r(2p-2n)+\sum_{j=1}^{r} b_j \geq r(2p-2n), \tag{4.5.1}$$

since $b_j\geq 0$. This gives the necessary condition for part (a) for the coset $\{\omega_1+\mathbb{Z}\Phi\}$. Before considering the remaining two cosets, towards addressing part (b), we consider when equality can hold in (4.5.1).

As in Section 4.2 we see that equality holds in (4.5.1) if and only if $N(\delta_j)=1$ and $\ell(u_j)=2n-2$, which forces $\lambda=\gamma_j=(p-2n+2)\omega_1$ for all j. Moreover, one obtains from the discussion above and Proposition 4.3.2 for $\lambda=(p-2n+2)\omega_1$ that

$$[H^i(G_1,H^0(\lambda))^{(-1)}]_\lambda \cong \begin{cases} H^0(\lambda) & \text{if } i=2p-2n\\ 0 & \text{if } 0<i<2p-2n. \end{cases} \tag{4.5.2}$$

Using the spectral sequence argument in the proof of Theorem 3.2.2 (see also the proof of [BNP, Lemma 5.4]), we can show that $\mathrm{H}^{r(2p-2n)}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(r)})=k$. We prove this by induction on r with the $r=1$ case being Theorem 4.5.1. Consider the Lyndon-Hochschild-Serre spectral sequence

$$\begin{aligned} E_2^{k,l} &= \mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)},\mathrm{H}^l(G_1,H^0(\lambda)))\\ &\cong \mathrm{Ext}^k_G(V(\lambda)^{(r-1)},\mathrm{H}^l(G_1,H^0(\lambda))^{(-1)}) \Rightarrow \mathrm{Ext}^{k+l}_G(V(\lambda)^{(r)},H^0(\lambda)). \end{aligned}$$

From the remarks in Section 2.8 and the discussion above, $E_2^{k,l}=0$ for $k+l<r(2p-2n)$. Furthermore, from (4.5.2), $E_2^{k,l}=0$ for $l<2p-2n$. Finally, if $E_2^{k,l}\neq 0$ and $k+l=r(2p-2n)$, then, from the above conclusion that $\gamma_j=\lambda$ for each j, we must have $l=2p-2n$. Hence, the $E_2^{(r-1)(2p-2n),2p-2n}$-term survives to E_∞ and is the only term to contribute in degree $r(2p-2n)$. Hence, by (4.5.2) and our inductive hypothesis,

$$\begin{aligned} \mathrm{Ext}^{r(2p-2n)}_G(V(\lambda)^{(r)},H^0(\lambda)) &\cong \mathrm{Ext}^{(r-1)(2p-2n)}_G(V(\lambda)^{(r-1)},\mathrm{H}^{2p-2n}(G_1,H^0(\lambda))^{(-1)})\\ &\cong \mathrm{Ext}^{(r-1)(2p-2n)}_G(V(\lambda)^{(r-1)},H^0(\lambda))\cong k. \end{aligned}$$

To complete the proof of (a) we need to consider the case that $\lambda=p\delta_0+u_0\cdot 0$ with $\delta_0=\delta_r\in\{\omega_{n-1}+\mathbb{Z}\Phi\}\cup\{\omega_n+\mathbb{Z}\Phi\}$. As above, $p\delta_j+u_j\cdot 0-\delta_{j-1}$ must lie in the positive root lattice. Since $u_j\cdot 0$ does, this implies that $p\delta_j-\delta_{j-1}$ must lie in the positive root lattice. When expressed as a sum of simple roots $\omega_{n-1}=\frac{1}{2}\alpha_1+\cdots$ (as does ω_n). Whereas, for $1\leq j\leq n-2$, $\omega_j=\alpha_1+\cdots$. Since $\delta_0\in\{\omega_{n-1}+\mathbb{Z}\Phi\}\cup\{\omega_n+\mathbb{Z}\Phi\}$, for $p\delta_1-\delta_0$ to lie in the positive root lattice, when expressed as a sum of fundamental weights, $p\delta_1$ must contain an odd number of copies of ω_{n-1} and ω_n in total. Since p is odd, this also holds for δ_1. Inductively, every δ_j has this property.

We may assume therefore that each δ_j contains at least one copy of ω_n or one copy of ω_{n-1}. Proceed as above, but let $N_{\alpha_n}(\gamma)$ and $N_{\alpha_{n-1}}(\gamma)$ denote the number of copies of α_n and α_{n-1}, respectively, appearing in γ. Set $N(\gamma)=\max\{N_{\alpha_n}(\gamma),N_{\alpha_{n-1}}(\gamma)\}$. Note that, for the weights γ that appear in what follows, both $N_{\alpha_n}(\gamma)$ and $N_{\alpha_{n-1}}(\gamma)$ are nonnegative. Again, a positive root contains at most one copy of α_n or α_{n-1}. Just as above, we get

$$l_j \quad\geq\quad 2pN_{\alpha_n}(\delta_j)-2N_{\alpha_n}(-u_j\cdot 0)-2N_{\alpha_n}(\delta_{j-1})+\ell(u_{j-1})$$

and the corresponding dual statement for α_{n-1}. By choosing the appropriate root we obtain

$$l_j \quad\geq\quad 2pN(\delta_j)-2N_{\alpha_n}(-u_j\cdot 0)-2N_{\alpha_n}(\delta_{j-1})+\ell(u_{j-1})$$

or

$$l_j \quad \geq \quad 2pN(\delta_j) - 2N_{\alpha_{n-1}}(-u_j \cdot 0) - 2N_{\alpha_{n-1}}(\delta_{j-1}) + \ell(u_{j-1}).$$

Either one will result in

$$l_j \quad \geq \quad 2pN(\delta_j) - 2N(-u_j \cdot 0) - 2N(\delta_{j-1}) + \ell(u_{j-1}).$$

From earlier arguments we know that $\ell(u_{j-1}) \geq N(-u_{j-1} \cdot 0)$. Hence

$$l_j \quad \geq \quad 2pN(\delta_j) - 2N(-u_j \cdot 0) - 2N(\delta_{j-1}) + N(-u_{j-1} \cdot 0).$$

Summing over j, one obtains

$$i = \sum_{j=1}^{r} l_j \geq \sum_{j=1}^{r} \left[(2p-2)N(\delta_j) - N(-u_j \cdot 0)\right].$$

Clearly, $N(-u_j \cdot 0) \leq N(-w_0 \cdot 0) = \frac{n(n-1)}{2}$. Moreover, we can say that $N(\delta_j) \geq \frac{n}{4}$. Substituting this gives

$$\begin{aligned} i &\geq r(2p-2)\left(\frac{n}{4}\right) - \frac{rn(n-1)}{2} \\ &= r\left(\frac{(p-1)(n)}{2} - \frac{n(n-1)}{2}\right) \\ &\geq r(2p-2n), \end{aligned}$$

where the last inequality follows as in the proof of Lemma 4.4.1. Thus part (a) follows. For $n \geq 5$, the last inequality is strict. Hence, $\lambda = (p-2n+2)\omega_1$ is the only dominant weight for which $\mathrm{H}^{r(2p-2n)}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. As in the proof of Theorem 4.5.1, since λ is minimal in its linkage class, part (b) follows. Similarly, for $n = 4$, by symmetry, part (b) follows. $\square$

5. Type E

5.1. Type E_6. Assume for this subsection that Φ is of type E_6 with $p > h = 12$ (so $p \geq 13$). The only dominant weights μ with $\langle \mu, \tilde{\alpha}^\vee \rangle < 2$ are ω_1 and ω_6. One concludes from Proposition 2.8.1 and Proposition 2.4.1 that $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ for all $0 < i < 2p-3$ unless there exists a weight λ of the form $p\omega_1 + w \cdot 0$ or of the form $p\omega_6 + w \cdot 0$ with $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $0 < i < 2p-3$.

LEMMA 5.1.1. *Suppose Φ is of type E_6, $p \geq 13$ and $\lambda \in X(T)_+$ is of the form $p\omega_1 + w \cdot 0$ or $p\omega_6 + w \cdot 0$ with $w \in W$. Assume in addition that $p \neq 13$, 19. Then $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ for all $0 < i < 2(p-1)$.*

PROOF. We prove the assertion for $\lambda = p\omega_1 + w \cdot 0, w \in W$. Let N denote the number of times that α_1 appears in $-w \cdot 0$ when written as a sum of simple roots. Note that all positive roots of Φ contain the simple root α_1 at most once. This implies that $N \leq \ell(w)$. Moreover, there are exactly 16 distinct positive roots containing α_1. Hence, $N \leq 16$.

Using $\omega_1 = 1/3(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)$, we note that $\lambda - \omega_1$, written as a sum of simple roots contains at least $4/3(p-1) - N$ copies of α_1. From Section 2.7 we know that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ and $i > 0$ imply that $\lambda - \omega_1$ is a sum of $(i - \ell(w))/2$ many positive roots. Note that this can only happen if $(p-1)$ is divisible by 3. Again using the fact that α_1 appears at most once in each positive root, one obtains the inequality:

$$\frac{4}{3}(p-1) - N \leq \frac{i - \ell(w)}{2}.$$

Solving for i yields

$$i \geq \frac{8}{3}(p-1) - 2N + \ell(w) \geq 2(p-1) + \frac{2}{3}(p-1) - N \geq 2(p-1) + \frac{2}{3}(p-1) - 16.$$

Note that equality holds if and only if $N = \ell(w) = 16$.

One obtains the desired claim $i \geq 2(p-1)$ for all primes except those of the form $p = 3t+1$ with $13 \leq p \leq 25$, i.e., the primes $p = 13$ and $p = 19$. $\square$

THEOREM 5.1.2. *Suppose Φ is of type E_6 and $p \geq 13$.*

(a) *If $p \neq 13$, then*

(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$*;*

(ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p),k)\neq 0$.

(b) *If* $p=13$, *then*

(i) $\mathrm{H}^i(G(\mathbb{F}_p),k)=0$ *for* $0<i<16$;

(ii) $\mathrm{H}^{16}(G(\mathbb{F}_p),k)\neq 0$.

Proof. For $p\neq 19$, part (a) follows immediately from Lemma 5.1.1, Proposition 2.4.1, and Theorem 3.2.2.

For the proof of part (b), set $p=13$. Part (i) follows from the proof of Lemma 5.1.1. Let W_I denote the subgroup of W generated by the simple reflections $s_{\alpha_2},...,s_{\alpha_6}$ and let w denote the distinguished representative of the left coset w_0W_I. Then $\ell(w)=16$ and $-w\cdot 0$ equals the sum of all positive roots in Φ that contain α_1, which equals the weight $12\omega_1$. Let $\lambda=p\omega_1+w\cdot 0=\omega_1$. Clearly,

$$\begin{aligned} k\cong \mathrm{Hom}_G(V(\lambda),H^0(\lambda)) &\cong \mathrm{Hom}_G(V(\lambda),\mathrm{ind}_B^G(S^0(\mathfrak{u}^*)\otimes\omega_1))\\ &\cong \mathrm{Hom}_G(V(\lambda),\mathrm{ind}_B^G(S^{(16-\ell(w))/2}(\mathfrak{u}^*)\otimes\omega_1))\\ &\cong \mathrm{Hom}_G(V(\lambda),\mathrm{H}^{16}(G_1,H^0(\lambda))^{(-1)})\\ &\cong \mathrm{H}^{16}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}). \end{aligned}$$

Since λ is the smallest dominant weight in its linkage class the assertion follows from the remarks in Section 2.5.

For $p=19$, part (a)(ii) follows from Theorem 3.2.2. It remains to show part (a)(i). If $\mathrm{H}^i(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})\neq 0$ for $i<35=2p-3$ and $\lambda\in X(T)_+$, then $\lambda=19\omega_1+w\cdot 0$ or $\lambda=19\omega_6+w\cdot 0$ for some $w\in W$. From the proof of Lemma 5.1.1, one can see that $i\geq 32$. Consider the case that $\lambda=19\omega_1+w\cdot 0$. The ω_6 case is dual and analogous. One can explicitly, with the aid of MAGMA, identify all $w\in W$ such that $\lambda\in X(T)_+$ and $\lambda-\omega_1$ lies in the positive root lattice. By considering the number of copies of α_1 appearing in $\lambda-\omega_1$ (as in the proof of Lemma 5.1.1), one can identify the least k such that $\lambda-\omega_1$ is a weight in $S^k(\mathfrak{u}^*)$, and hence the least possible value of i. The three weights which can give a value of $i<35$ are listed in the following table along with the minimum possible value of k.

λ	$\ell(w)$	k	$i=2k+\ell(w)$
$7\omega_1+\omega_4$	14	10	34
$7\omega_1+\omega_2$	15	9	33
$7\omega_1$	16	8	32

For these weights, one can use MAGMA to explicitly compute

$$\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot\lambda-\omega_1)$$

in order to apply Proposition 2.7.1. For $\lambda=7\omega_1$, one finds that in fact

$$\dim\mathrm{H}^{32}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=\sum_{u\in W}(-1)^{\ell(u)}P_8(u\cdot\lambda-\omega_1)=0$$

and

$$\dim\mathrm{H}^{34}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=\sum_{u\in W}(-1)^{\ell(u)}P_9(u\cdot\lambda-\omega_1)=0.$$

So, for $\lambda=7\omega_1$, we have $i\geq 36$.

For $\lambda=7\omega_1+\omega_2$ one finds

$$\dim\mathrm{H}^{33}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=\sum_{u\in W}(-1)^{\ell(u)}P_9(u\cdot\lambda-\omega_1)=0.$$

Therefore, $i\geq 35$ in this case.

Finally, for $\lambda=7\omega_1+\omega_4$, one finds

$$\dim\mathrm{H}^{34}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=\sum_{u\in W}(-1)^{\ell(u)}P_{10}(u\cdot\lambda-\omega_1)=0.$$

Therefore, $i\geq 36$ in this case and the claim follows. □

We now consider the situation for arbitrary r. Sharp vanishing can be obtained for primes about twice the Coxeter number.

THEOREM 5.1.3. *Suppose Φ is of type E_6 and $p \geq 13$.*

(a) *If $p \neq 13, 19$ or $p \neq 17$ when r is even, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$*;*
  (ii) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.
(b) *If $p = 13$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < 16r$*;*
  (ii) $\mathrm{H}^{16r}(G(\mathbb{F}_q), k) \neq 0$.
(c) *If $p = 17$ and r is even, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < 27r$*;*
  (ii) $\mathrm{H}^{31r}(G(\mathbb{F}_q), k) \neq 0$.
(d) *If $p = 19$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < 32r$*;*
  (ii) $\mathrm{H}^{35r}(G(\mathbb{F}_q), k) \neq 0$.

PROOF. The validity of parts (a)(ii), (c)(ii), and (d)(ii) follows from Theorem 3.2.2. For part (a)(i), we need to show that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ for all dominant weights λ and all $0 < i < r(2p-3)$. We argue along lines similar to that of the proof of Theorem 4.5.2. The root lattice $\mathbb{Z}\Phi$ has three cosets within $X(T)$: $\mathbb{Z}\Phi$, $\{\omega_1 + \mathbb{Z}\Phi\}$, and $\{\omega_6 + \mathbb{Z}\Phi\}$. If λ is a weight in the root lattice, claim (a)(i) follows from Lemma 3.2.1(a).

Assume that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$ for some $i > 0$ and apply Proposition 2.8.2. From above, we may assume that $\lambda = p\delta_0 + u_0 \cdot 0$ with $\delta_0 = \delta_r \in \{\omega_1 + \mathbb{Z}\Phi\} \cup \{\omega_6 + \mathbb{Z}\Phi\}$. For each $1 \leq j \leq r$, in order to have $\mathrm{Ext}_G^{l_j}(V(\gamma_j)^{(1)}, H^0(\gamma_{j-1})) \neq 0$, then $\gamma_j - \delta_{j-1} = p\delta_j + u_j \cdot 0 - \delta_{j-1}$ must be a weight in $S^{\frac{l_j - \ell(u_{j-1})}{2}}(\mathfrak{u}^*)$. This implies that $p\delta_j - \delta_{j-1}$ must lie in the positive root lattice. Since $\delta_r \in \{\omega_1 + \mathbb{Z}\Phi\} \cup \{\omega_6 + \mathbb{Z}\Phi\}$, we necessarily have $p\delta_r \in \{\omega_1 + \mathbb{Z}\Phi\} \cup \{\omega_6 + \mathbb{Z}\Phi\}$. Since $p\delta_r - \delta_{r-1} \in \mathbb{Z}\Phi$, it then follows that $\delta_{r-1} \in \{\omega_1 + \mathbb{Z}\Phi\} \cup \{\omega_6 + \mathbb{Z}\Phi\}$. Inductively one concludes that $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\} \cup \{\omega_6 + \mathbb{Z}\Phi\}$ for all j.

Before continuing, we investigate this condition on δ_j a bit further. Recall that $\omega_1 = \frac{4}{3}\alpha_1 + \cdots + \frac{2}{3}\alpha_6$ and $\omega_6 = \frac{2}{3}\alpha_1 + \cdots + \frac{4}{3}\alpha_6$. Suppose that $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\}$ and $\delta_{j-1} \in \{\omega_1 + \mathbb{Z}\Phi\}$. In order for $p\delta_j - \delta_{j-1}$ to lie in the root lattice, $\frac{4}{3}p - \frac{4}{3} = \frac{4}{3}(p-1)$ would need to be an integer. In other words, $p-1$ must be divisible by 3. The same argument holds if we assume that both δ_j and δ_{j-1} lie in $\{\omega_6 + \mathbb{Z}\Phi\}$. On the other hand, suppose that $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\}$ and $\delta_{j-1} \in \{\omega_6 + \mathbb{Z}\Phi\}$ (or vice versa). Then $\frac{4}{3}p - \frac{2}{3} = \frac{2}{3}(2p-1)$ (or $\frac{2}{3}p - \frac{4}{3} = \frac{2}{3}(p-2)$, respectively) must be an integer which implies that $p-2$ is divisible by 3. Since p is a prime greater than three, either $p-1$ is divisible by 3 or $p-2$ is divisible by 3. Summarizing, if $3|(p-1)$, then either each $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\}$ or each $\delta_j \in \{\omega_6 + \mathbb{Z}\Phi\}$. We refer to this as the "consistent" case. Whereas, if $3|(p-2)$, then we have an "alternating" situation with the δ_js alternately lying in $\{\omega_1 + \mathbb{Z}\Phi\}$ or $\{\omega_6 + \mathbb{Z}\Phi\}$. Note further that since $\delta_0 = \delta_r$, the alternating case can only occur if r is *even*.

Consider first the consistent case (when $3|(p-1)$). Suppose without loss of generality that each $\delta_j \in \{\omega_1 + \mathbb{Z}\Phi\}$. For a weight γ, when expressed as a sum of simple roots, let $N(\gamma)$ denote the number of copies of α_1 that appear. Since a positive root contains at most one copy of α_1, we have

$$\frac{l_j - \ell(u_{j-1})}{2} \geq pN(\delta_j) - N(-u_j \cdot 0) - N(\delta_{j-1}).$$

Rewriting this and using the fact that (see Observation 2.2.1) $\ell(u_{j-1}) \geq N(-u_{j-1} \cdot 0)$ gives

$$\begin{aligned} l_j &\geq 2pN(\delta_j) - 2N(-u_j \cdot 0) - 2N(\delta_{j-1}) + \ell(u_{j-1}) \\ &\geq 2pN(\delta_j) - 2N(-u_j \cdot 0) - 2N(\delta_{j-1}) + N(-u_{j-1} \cdot 0). \end{aligned}$$

Therefore,

$$\begin{aligned} i = \sum_{j=1}^{r} l_j &\geq \sum_{j=1}^{r}(2pN(\delta_j) - 2N(-u_j\cdot 0) - 2N(\delta_{j-1}) + N(-u_{j-1}\cdot 0)) \\ &= \sum_{j=1}^{r}((2p-2)N(\delta_j) - N(-u_j\cdot 0)). \end{aligned}$$

There are only 16 positive roots which contain an α_1. Hence, $N(-u_j \cdot 0) \leq 16$. Since $N(\delta_j) \geq \frac{4}{3}$, we get

$$i \geq \sum_{j=1}^{r}\left(\frac{4}{3}(2p-2) - 16\right) = r\left(\frac{4}{3}(2p-2) - 16\right) = r\left(2p-2+\frac{1}{3}(2p-2)-16\right).$$

For $p \geq 25$, we get $i \geq r(2p-2)$ as desired. Note that for $p = 17$ and $p = 23$, $3 \nmid (p-1)$, and so the only "bad" cases are $p = 13$ and $p = 19$. For $p = 13$, we conclude that $i \geq 16r$, and for $p = 19$, we conclude that $i \geq 32r$.

Now consider the alternating case (which requires $p-2$ being divisible by 3). Analogous to the proof of Theorem 4.5.2 for the type D_n case, for a weight γ, let $N_{\alpha_1}(\gamma)$ (or $N_{\alpha_6}(\gamma)$) denote the coefficient of α_1 (or α_6) when γ is expressed as a sum of simple roots. And then set $N(\gamma) = \max\{N_{\alpha_1}(\gamma), N_{\alpha_6}(\gamma)\}$ (where the max is considered only in cases where the quantities involved are nonnegative). Then we reach the same conclusion on i as above. In this case, $p = 13$ and $p = 19$ cannot occur. Moreover, $p = 17$ and $p = 23$ are potentially "bad." However, for $p = 23$, since i is an integer, we still conclude that $i \geq r(2p-3)$ as needed. For $p = 17$, we conclude that $i \geq 27r$.

That completes the proof of all parts except for part (b)(ii) with $p = 13$. This follows inductively from the $r = 1$ case by using the the spectral sequence argument as in the proofs of Theorem 3.2.2 and Theorem 4.5.2. □

For $p = 17$ when r is even and $p = 19$, the theorem does not give a sharp vanishing bound.

5.2. Type E_7. Assume for this subsection that Φ is of type E_7 with $p > h = 18$ (so $p \geq 19$.) The only dominant weight μ with $\langle \mu, \tilde{\alpha}\rangle < 2$ is ω_7. Again we conclude from Proposition 2.8.1 and Proposition 2.4.1 that $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ for all $0 < i < 2p-3$ unless there exists a weight λ of the form $p\omega_7 + w\cdot 0$ with $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $0 < i < 2p-3$.

LEMMA 5.2.1. *Suppose Φ is of type E_7, $p \geq 19$ and $\lambda \in X(T)_+$ is of the form $p\omega_7 + w\cdot 0$ with $w \in W$. Assume in addition that $p \neq 19$, 23. Then $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for all $0 < i < 2(p-1)$.*

PROOF. Assume $\lambda = p\omega_7 + w\cdot 0, w \in W$. Let N denote the number of times that α_7 appears in $-w\cdot 0$ when written as a sum of simple roots. Note that all positive roots of Φ contain the simple root α_7 at most once. This implies that $N \leq \ell(w)$. Moreover, there are exactly 27 distinct positive roots containing α_7. Hence, $N \leq 27$.

When writing ω_7 as a sum of simple roots the coefficient for α_7 is 3/2. Therefore $\lambda - \omega_7$, written as a sum of simple roots contains at least $3/2(p-1) - N$ copies of α_7. From Section 2.7, we know that $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ and $i > 0$ imply that $\lambda - \omega_1$ is a sum of $(i - \ell(w)/2$ many positive roots. Using the the fact that α_7 appears at most once in each positive root, one obtains the inequality:

$$\frac{3}{2}(p-1) - N \leq \frac{i-\ell(w)}{2}.$$

Solving for i yields

$$i \geq 3(p-1) - 2N + \ell(w) \geq 2(p-1) + p - 1 - N \geq 2(p-1) + p - 1 - 27.$$

Note that equality holds if and only if $N = \ell(w) = 27$.

Hence, $i \geq 2(p-1)$ for all primes except for $18 < p \leq 28$, i.e., the primes $p = 19$ and $p = 23$. □

THEOREM 5.2.2. *Suppose Φ is of type E_7 and $p \geq 19$.*

(a) *If $p \neq 19, 23$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$*;*
  (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

(b) *If* $p = 19$, *then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 27$;
  (ii) $\mathrm{H}^{27}(G(\mathbb{F}_p), k) \neq 0$.
(c) *If* $p = 23$, *then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 39$;
  (ii) $\mathrm{H}^{39}(G(\mathbb{F}_p), k) \neq 0$.

Proof. Part (a) follows from Lemma 5.2.1, Proposition 2.4.1, and Theorem 3.2.2.

For the proof of part (b), set $p = 19$. Part (i) follows from the proof of Lemma 5.2.1. Let W_I denote the subgroup of W generated by the simple reflections $s_{\alpha_1}, ..., s_{\alpha_6}$ and let w denote the distinguished representative of the left coset w_0W_I. Then $\ell(w) = 27$ and $-w \cdot 0$ equals the sum of all positive roots in Φ that contain α_7, which equals the weight $18\omega_7$. Let $\lambda = p\omega_7 + w \cdot 0 = \omega_7$. Using the same argument as for E_6, we obtain $\mathrm{H}^{27}(G_1, H^0(\lambda))^{(-1)} \cong H^0(\lambda)$ and hence $\mathrm{H}^{27}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong k$. Again λ is the smallest dominant weight in its linkage class and the assertion follows from the remarks in Section 2.5.

For part (c), set $p = 23$. If $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $i < 43 = 2p - 3$ and $\lambda \in X(T)_+$, then $\lambda = 23\omega_7 + w \cdot 0$ for some $w \in W$. From the proof of Lemma 5.2.1, one can see that $i \geq 39$. One can explicitly, with the aid of MAGMA, identify all $w \in W$ such that $\lambda \in X(T)_+$ and $\lambda - \omega_7$ lies in the positive root lattice. By considering the number of copies of α_7 appearing in $\lambda - \omega_7$ (as in the proof of Lemma 5.2.1), one can identify the least k such that $\lambda - \omega_7$ is a weight in $S^k(\mathfrak{u}^*)$, and hence the least possible value of i. The four weights which can give a value of $i < 43$ are listed in the following table along with the minimum possible value of k.

λ	$\ell(w)$	k	$i = 2k + \ell(w)$
$5\omega_7 + \omega_4$	24	9	42
$5\omega_7 + \omega_3$	25	8	41
$5\omega_7 + \omega_1$	26	7	40
$5\omega_7$	27	6	39

For these weights, one can use MAGMA to explicitly compute

$$\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot \lambda - \omega_7)$$

in order to apply Proposition 2.7.1. For $\lambda = 5\omega_7$, one finds that in fact

$$\dim \mathrm{H}^{39}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_6(u \cdot \lambda - \omega_7) = 1.$$

Since there are no weights less than λ which can give cohomology in degree 40, $\mathrm{H}^{39}(G(\mathbb{F}_p), k) \neq 0$. □

We next consider the situation for arbitrary r.

Theorem 5.2.3. *Suppose* Φ *is of type* E_7 *and* $p \geq 19$.
(a) *If* $p \neq 19, 23$, *then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p - 3)$;
  (ii) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.
(b) *If* $p = 19$, *then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < 27r$;
  (ii) $\mathrm{H}^{27r}(G(\mathbb{F}_q), k) \neq 0$.
(c) *If* $p = 23$, *then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < 39r$;
  (ii) $\mathrm{H}^{39r}(G(\mathbb{F}_q), k) \neq 0$.

Proof. The validity of part (a)(ii) follows from Theorem 3.2.2. For part (a)(i), we need to show that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ for all dominant weights λ and all $0 < i < r(2p - 3)$. An argument similar to that in the proof of Theorem 5.1.3 works here as well. The root lattice $\mathbb{Z}\Phi$ has two cosets within $X(T)$: $\mathbb{Z}\Phi$ and $\{\omega_7 + \mathbb{Z}\Phi\}$. If λ is a weight in the root lattice claim (a)(i) follows from Lemma 3.2.1(a).

Consider then the case that $\lambda \in \{\omega_7 + \mathbb{Z}\Phi\}$ and apply Proposition 2.8.2. As before, one finds that each $\delta_j \in \{\omega_7 + \mathbb{Z}\Phi\}$. Further, if we let $N(\gamma)$ denote the coefficient of α_7, when γ is expressed as a sum of simple roots, then we again conclude that

$$i \geq \sum_{j=1}^{r}((2p-2)N(\delta_j) - N(-u_j \cdot 0)).$$

Here, since $\omega_7 = \alpha_1 + \cdots + \frac{3}{2}\alpha_7$, we have $N(\delta_j) \geq \frac{3}{2}$. Furthermore, there are 27 positive roots which contain an α_7, and so $N(-u_j \cdot 0) \leq 27$. Therefore, we get

$$i \geq r\left(\frac{3}{2}(2p-2) - 27\right) = r\left(2p - 3 + p - 27\right).$$

For $p \geq 27$, we have $i \geq r(2p-3)$ as needed, which completes part (a).

For $p = 19$ and $p = 23$, we conclude only that $i \geq 27r$ or $i \geq 39r$, respectively, which gives parts (b)(i) and (c)(i). Parts (b)(ii) and (c)(ii) again follows inductively from the $r = 1$ case by the spectral sequence argument in Theorem 3.2.2 and Theorem 4.5.2. □

5.3. Type E_8. Assume for this subsection that Φ is of type E_8 with $p > h = 30$ (so $p \geq 31$). Here the weight lattice and root lattice always coincide. From Corollary 3.3.1 we obtain the following.

THEOREM 5.3.1. *Suppose Φ is of type E_8 and $p \geq 31$. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$*;*
(b) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.

6. Type B_n, $n \geq 3$

Assume throughout this section that Φ is of type B_n, $n \geq 3$, and that $p > h = 2n$. Note that type B_2 is equivalent to type C_2 which was discussed in [BNP]. However, for certain inductive arguments, at points we will allow $n = 1, 2$. Following Section 2, our goal is to find the least $i > 0$ such that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some λ. From Proposition 2.8.1, we know that $i \geq p - 2$.

6.1. Restrictions. Suppose that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i > 0$ and $\lambda = p\mu + w \cdot 0$ with $\mu \in X(T)_+$ and $w \in W$. In this case, the longest root $\tilde{\alpha} = \omega_2$. From Proposition 2.8.1, $i \geq (p-1)\langle \mu, \tilde{\alpha}^\vee \rangle - 1$.

For a fundamental dominant weight ω_j,

$$\langle \omega_j, \tilde{\alpha}^\vee \rangle = \begin{cases} 1 & \text{if } j = 1, n \\ 2 & \text{if } 2 \leq j \leq n-1. \end{cases}$$

Therefore, if $\mu \neq \omega_1, \omega_n$, we will have $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 2$ and $i \geq 2p - 3$.

The following lemma shows that if n is sufficiently large, and $\lambda = p\omega_n + w \cdot 0$, then one also has $i \geq 2p-3$. In fact strictly greater.

LEMMA 6.1.1. *Suppose Φ is of type B_n with $n \geq 7$ and $p > 2n$. Suppose $\lambda = p\omega_n + w \cdot 0 \in X(T)_+$ with $w \in W$ and $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Then $i > 2p - 3$.*

PROOF. Following the discussion in Section 2.7, $\lambda - \omega_n = (p-1)\omega_n + w \cdot 0$ must be a weight of $S^j(\mathfrak{u}^*)$ for $j = \frac{i - \ell(w)}{2}$. Recall that $2\omega_n = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n$. Consider the decomposition of $-w \cdot 0$ into a sum of distinct positive roots (cf. Observation 2.2.1). Write $\ell(w) = a + b + c$ where a is the number of positive roots in this decomposition which contain $2\alpha_n$, b is the number of roots in this decomposition which contain α_n, and c is the number of roots in this decomposition that do not contain α_n. Then $\lambda - \omega_n$ contains

$$\left(\frac{p-1}{2}\right)n - 2a - b$$

copies of α_n. Since any root contains at most 2 copies of α_n, we have

$$\frac{i - \ell(w)}{2} = j \geq \frac{1}{2}\left(\left(\frac{p-1}{2}\right)n - 2a - b\right).$$

Replacing $\ell(w)$ by $a+b+c$ and simplifying gives

$$i \geq \left(\frac{p-1}{2}\right) n - a + c.$$

The total number of positive roots which contain $2\alpha_n$ is $n(n-1)/2$. Hence, $a \leq n(n-1)/2$ and $c \geq 0$. So we have

$$i \geq \left(\frac{p-1}{2}\right) n - \left(\frac{n-1}{2}\right) n = \left(\frac{p-n}{2}\right) n = 2p + \left(\frac{n}{2} - 2\right) p - \frac{n^2}{2}. \tag{6.1.1}$$

Finally, using the assumption that $p \geq 2n+1$, one finds

$$i \geq 2p + \left(\frac{n}{2} - 2\right)(2n+1) - \frac{n^2}{2} = 2p + \frac{n^2}{2} - \frac{7}{2}n - 2.$$

For $n \geq 7$, we have $i \geq 2p-2$ as claimed. □

For $3 \leq n \leq 6$, the lemma is in fact false. These cases will be discussed in Sections 6.3 - 6.6.

6.2. The case of ω_1. In this section we investigate the case that $\lambda = p\omega_1 + w \cdot 0$. Throughout this section Φ is of type B_n. In order to make use of some inductive arguments we allow $n \geq 1$. We will frequently switch between the bases consisting of the simple roots $\{\alpha_1, \dots, \alpha_n\}$, the fundamental weights $\{\omega_1, \dots, \omega_n\}$, and the canonical basis $\{\epsilon_1, \dots, \epsilon_n\}$ of $\mathbb{R}^n$. Following [Hum] we have $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $1 \leq i \leq n-1$, and $\alpha_n = \epsilon_n$. Note that $\epsilon_1 = \alpha_0$ is the maximal short root. The fundamental weights are $\omega_j = \epsilon_1 + \cdots + \epsilon_j$, for $1 \leq j \leq n-1$, and $\omega_n = 1/2(\epsilon_1 + \cdots + \epsilon_n)$. In particular, $\omega_1 = \epsilon_1$.

Definition 6.2.1. For Φ of type B_n, with $n \geq 1$, we define

$$P(m,k,j,n) := \begin{cases} \sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot (m\epsilon_1 + (\epsilon_1 + \cdots + \epsilon_j))) & \text{if } m \geq 0, k \geq 0, \\ & \quad \text{and } 1 \leq j \leq n, \\ 1 & \text{if } m = -1, k = 0, \\ & \quad \text{and } j = 1, \\ 0 & \text{else;} \end{cases}$$

$$T(m,k,j,n) := \begin{cases} \dim \operatorname{Hom}_G(V(m\epsilon_1 + (\epsilon_1 + \cdots + \epsilon_j)), H^0(G/B, S^k(\mathfrak{u}^*)) \otimes H^0(\epsilon_1)) & \\ \qquad\qquad \text{if } m \geq 0, k \geq 0, \text{ and } 1 \leq j \leq n, & \\ 0 \qquad\qquad \text{else.} & \end{cases}$$

Note that for $p > 2n$,

$$P(m,k,j,n) = \dim \operatorname{Hom}_G(V(m\epsilon_1 + (\epsilon_1 + ... + \epsilon_j)), H^0(G/B, S^k(\mathfrak{u}^*))),$$

which equals the multiplicity of $H^0(m\epsilon_1 + (\epsilon_1 + \cdots + \epsilon_j))$ in a good filtration of $H^0(G/B, S^k(\mathfrak{u}^*))$ (cf. [AJ, 3.8]). In particular, $P(m,k,j,n) \geq 0$ for all m, k, j, and n.

Lemma 6.2.2. *Suppose Φ is of type B_n with $n \geq 1$ and $p > 2n$. If $m \geq 0$, $k \geq 0$, and $1 \leq j \leq n$, then*

(a) $\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot (m\epsilon_1 + (\epsilon_1 + ... + \epsilon_j)) - \epsilon_1) = P(m-1,k,j,n) + P(m,k,j-1,n-1)$;
(b) $\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot m\epsilon_1 + \epsilon_1) = P(m,k,1,n) - P(m,k,1,n-1)$;
(c) $P(m,k,j,n) = 0$ *whenever* $k < m+1$;
(d) $T(m,k,j,n) = \sum_{i=1}^{2n} P(m-1,k-i+1,j,n) + \sum_{i=1}^{2n} P(m,k-i+1,j-1,n-1) + P(m,k-n,j,n)$;
(e) *for* $n \geq 2$ *and* $1 \leq j \leq n-1$,
$T(m,k,j,n) \geq P(m-1,k,j,n) + P(m,k,j+1,n) + P(m,k,j-1,n)$;
(f) $T(m,k,n,n) \geq P(m-1,k,n,n) + P(m,k,n,n) + P(m,k,n-1,n)$;
(g) *for* $l \geq 0$, $P(2l,k,1,n) = P(2l-1,k-n,1,n)$;
(h) *for* $l \geq 0$, $\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot (2l+1)\epsilon_1 + \epsilon_1) = P(2l,k-n,1,n)$.

Proof. (a) $P(m-1,k,j,n) = \sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)))$. The expression $u\cdot((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)) = u((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))+u\cdot 0$ will be a sum of positive roots only if either $u(\epsilon_1)=\epsilon_1$ or $u(\epsilon_2)=\epsilon_1$. If u is of the second type, then us_{α_1} stabilizes ϵ_1. Setting $v=us_{\alpha_1}$, one obtains

$$\begin{aligned} P(m-1,k,j,n) &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))) \\ &\quad+\sum_{\{u\in W|u(\epsilon_2)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))) \\ &= \sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1) \\ &\quad-\sum_{v\in W}(-1)^{\ell(v)}P_k(v\cdot(m\epsilon_2+(\epsilon_2+\cdots+\epsilon_j))). \end{aligned}$$

The second term is just $P(m,k,j-1,n-1)$ as claimed. Note that the formula also holds for $m=0$.

(b) The expression $\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1)$ will be a sum of positive roots only if either $u(\epsilon_1)=\epsilon_1$ or $u(\epsilon_2)=\epsilon_1$. Arguing as above one obtains

$$\begin{aligned} \sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1) &= \sum_{\{u\in W|u(\epsilon_1)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1) \\ &\quad+\sum_{\{u\in W|u(\epsilon_2)=\epsilon_1\}}(-1)^{\ell(u)}P_k(u\cdot m\epsilon_1+\epsilon_1) \\ &= \sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot(m\epsilon_1+\epsilon_1)) \\ &\quad-\sum_{v\in W}(-1)^{\ell(v)}P_k(v\cdot(m\epsilon_2+\epsilon_2)). \end{aligned}$$

The first term is $P(m,k,1,n)$ and the second term is just $P(m,k,1,n-1)$.

(c) Assume that $0\le k<m+1$. Part (a) implies that

$$P(m,k,j,n)\le\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1).$$

Note that $u\cdot((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1$ is a sum of positive roots if and only if $u(\epsilon_1)=\epsilon_1$. If $u(\epsilon_1)=\epsilon_1$ then $u\cdot((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1=(m+1)\epsilon_1+u\cdot(\epsilon_2+\cdots+\epsilon_j)$ and $-u\cdot(\epsilon_2+\cdots+\epsilon_j)$, written as a sum of simple roots, contains no α_1. However, $(m+1)\epsilon_1$, written as a sum of simple roots, contains exactly $m+1$ copies of α_1. Each positive root of Φ contains at most one copy of α_1. Therefore at least $m+1$ positive roots are needed to obtain the weight $u\cdot((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1$. One concludes that $P_k(u\cdot((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))-\epsilon_1)=0$ and the assertion follows.

(d) For a simple root α, let P_α denote the minimal parabolic subgroup corresponding to α, and let $\mathfrak{u}_\alpha$ denote the Lie algebra of the unipotent radical of P_α. From the short exact sequence

$$0\to\alpha\to\mathfrak{u}^*\to\mathfrak{u}_\alpha^*\to 0$$

one obtains the Koszul resolution

$$0\to S^{k-1}(\mathfrak{u}^*)\otimes\alpha\to S^k(\mathfrak{u}^*)\to S^k(\mathfrak{u}_\alpha^*)\to 0.$$

Tensoring with a weight μ yields

$$0\to S^{k-1}(\mathfrak{u}^*)\otimes\alpha\otimes\mu\to S^k(\mathfrak{u}^*)\otimes\mu\to S^k(\mathfrak{u}_\alpha^*)\otimes\mu\to 0.$$

Induction from B to G yields the long exact sequence

$$\cdots\to H^i(G/B,S^{k-1}(\mathfrak{u}^*)\otimes\alpha\otimes\mu)\to H^i(G/B,S^k(\mathfrak{u}^*)\otimes\mu)\to H^i(G/B,S^k(\mathfrak{u}_\alpha^*)\otimes\mu)\to\cdots. \tag{6.2.1}$$

We apply (6.2.1) with $\alpha=\alpha_j=\epsilon_j-\epsilon_{j+1}$ and $\mu=-\epsilon_j$, where $1\le j\le n-1$, giving

$$\cdots\to H^i(G/B,S^{k-1}(\mathfrak{u}^*)\otimes-\epsilon_{j+1})\to H^i(G/B,S^k(\mathfrak{u}^*)\otimes-\epsilon_j)\to H^i(G/B,S^k(\mathfrak{u}_\alpha^*)\otimes-\epsilon_j)\to\cdots. \tag{6.2.2}$$

Note that, $\langle -\epsilon_j, \alpha_j^\vee\rangle = -1$, forces $H^i(P_\alpha/B, -\epsilon_j) = 0$ for all i. The spectral sequence

$$H^r(G/P_\alpha, S^k(\mathfrak{u}_\alpha^*)) \otimes H^s(P_\alpha/B, \mu) \Rightarrow H^{r+s}(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes \mu)$$

yields $H^i(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes -\epsilon_j) = 0$ for all i. Therefore, from (6.2.2), one obtains for $1 \le j \le n-1$ and $i \ge 0$

$$H^i(G/B, S^{k-1}(\mathfrak{u}^*) \otimes -\epsilon_{j+1}) \cong H^i(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_j). \tag{6.2.3}$$

Iterating this process yields

$$H^i(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_i) \cong H^i(G/B, S^{k-n+i}(\mathfrak{u}^*) \otimes -\epsilon_n). \tag{6.2.4}$$

Note that if $k < n - i$, the right hand side is identically zero, and the isomorphism still holds.

Next we apply (6.2.1) with $\alpha = \alpha_n = \epsilon_n$ and $\mu = -\epsilon_n$ in order to obtain

$$\cdots \to H^i(G/B, S^{k-1}(\mathfrak{u}^*)) \to H^i(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_n) \to H^i(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes -\epsilon_n) \to \cdots. \tag{6.2.5}$$

Here $\langle -\epsilon_n, \alpha_n^\vee\rangle = -2$. Using the spectral sequence as above, one obtains

$$\begin{aligned} H^i(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes -\epsilon_n) &\cong H^{i-1}(G/P_\alpha, S^k(\mathfrak{u}_\alpha^*)) \otimes H^1(P_\alpha/B, -\epsilon_n) \\ &\cong H^{i-1}(G/P_\alpha, S^k(\mathfrak{u}_\alpha^*)) \cong H^{i-1}(G/B, S^k(\mathfrak{u}_\alpha^*)). \end{aligned} \tag{6.2.6}$$

Since $H^0(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes -\epsilon_n) = 0$, one obtains via by (6.2.5),

$$H^0(G/B, S^{k-1}(\mathfrak{u}^*)) \cong H^0(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_n). \tag{6.2.7}$$

From $H^i(G/B, S^k(\mathfrak{u}^*)) = 0$ for $i \ge 1$, using (6.2.5), one concludes

$$H^i(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_n) \cong H^{i-1}(G/B, S^k(\mathfrak{u}_\alpha^*)) \text{ for } i \ge 1. \tag{6.2.8}$$

Next, we apply (6.2.1) with $\alpha = \alpha_n = \epsilon_n$ and $\mu = 0$ in order to obtain

$$\cdots \to H^i(G/B, S^{k-1}(\mathfrak{u}^*) \otimes \epsilon_n) \to H^i(G/B, S^k(\mathfrak{u}^*)) \to H^i(G/B, S^k(\mathfrak{u}_\alpha^*)) \to \cdots. \tag{6.2.9}$$

From $H^i(G/B, S^k(\mathfrak{u}^*)) = 0$ and $H^i(G/B, S^k(\mathfrak{u}^*) \otimes \epsilon_n) = 0$ for $i \ge 1$ [KLT, 2.8], one concludes

$$0 \to H^0(G/B, S^{k-1}(\mathfrak{u}^*) \otimes \epsilon_n) \to H^0(G/B, S^k(\mathfrak{u}^*)) \to H^0(G/B, S^k(\mathfrak{u}_\alpha^*)) \to 0, \tag{6.2.10}$$

and

$$H^i(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_n) \cong H^{i-1}(G/B, S^k(\mathfrak{u}_\alpha^*)) = 0 \text{ for } i \ge 2. \tag{6.2.11}$$

Similarly, apply (6.2.1) with $\alpha = \alpha_j = \epsilon_{j-1} - \epsilon_j$ and $\mu = \epsilon_j$, where $2 \le j \le n$, to obtain

$$\cdots \to H^i(G/B, S^{k-1}(\mathfrak{u}^*) \otimes \epsilon_{j-1}) \to H^i(G/B, S^k(\mathfrak{u}^*) \otimes \epsilon_j) \to H^i(G/B, S^k(\mathfrak{u}_\alpha^*) \otimes \epsilon_j) \to \cdots.$$

As before this yields

$$H^i(G/B, S^{k-1}(\mathfrak{u}^*) \otimes \epsilon_{j-1}) \cong H^i(G/B, S^k(\mathfrak{u}^*) \otimes \epsilon_j). \tag{6.2.12}$$

Iterating this process results in

$$H^i(G/B, S^{k-i+1}(\mathfrak{u}^*) \otimes \epsilon_1) \cong H^i(G/B, S^k(\mathfrak{u}^*) \otimes \epsilon_i). \tag{6.2.13}$$

Again, this isomorphism holds even if $k < i - 1$ (when the left side is identically zero).

For any finite B-module M, we denote its Euler characteristic by

$$\chi(M) = \sum_{i\ge 0} (-1)^i \operatorname{ch} H^i(G/B, M).$$

From the above, one obtains

$$
\begin{aligned}
&\operatorname{ch}(H^0(G/B,S^k(\mathfrak{u}^*))\otimes H^0(\epsilon_1)) = \chi(S^k(\mathfrak{u}^*)\otimes H^0(\epsilon_1))\\
&= \sum_{i=1}^{n}\chi(S^k(\mathfrak{u}^*)\otimes\epsilon_i)+\chi(S^k(\mathfrak{u}^*))+\sum_{i=1}^{n}\chi(S^k(\mathfrak{u}^*)\otimes-\epsilon_i)\\
&= \sum_{i=1}^{n}\chi(S^{k-i+1}(\mathfrak{u}^*)\otimes\epsilon_1)+\chi(S^k(\mathfrak{u}^*))+\sum_{i=1}^{n}\chi(S^{k-i+1}(\mathfrak{u}^*)\otimes-\epsilon_n)\ \text{(by 6.2.4, 6.2.13)}\\
&= \sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes\epsilon_1)+\operatorname{ch}H^0(G/B,S^k(\mathfrak{u}^*))\ \text{(by [KLT, 2.8])}\\
&\quad+\sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes-\epsilon_n)-\sum_{i=1}^{n}\operatorname{ch}H^1(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes-\epsilon_n)\\
&\qquad\qquad\text{(by 6.2.8, 6.2.11)}\\
&= \sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes\epsilon_1)+\operatorname{ch}H^0(G/B,S^k(\mathfrak{u}^*))\\
&\quad+\sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i}(\mathfrak{u}^*))+\sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i-n+1}(\mathfrak{u}^*)\otimes\epsilon_1)\\
&\quad-\sum_{i=1}^{n}\operatorname{ch}H^0(G/B,S^{k-i+1}(\mathfrak{u}^*))\ \text{(by 6.2.7, 6.2.8, 6.2.10, 6.2.13)}\\
&= \sum_{i=1}^{2n}\operatorname{ch}H^0(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes\epsilon_1)+\operatorname{ch}H^0(G/B,S^{k-n}(\mathfrak{u}^*)).
\end{aligned}
$$

The last equality yields

$$
\begin{aligned}
T(m,k,j,n) &= \dim\operatorname{Hom}_G(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)),H^0(G/B,S^k(\mathfrak{u}^*))\otimes H^0(\epsilon_1))\\
&= \sum_{i=1}^{2n}\dim\operatorname{Hom}_G(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)),H^0(G/B,S^{k-i+1}(\mathfrak{u}^*)\otimes\epsilon_1))\\
&\quad+\dim\operatorname{Hom}_G(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)),H^0(G/B,S^{k-n}(\mathfrak{u}^*)).
\end{aligned}
$$

The assertion follows now from part (a).

(e) A direct computation shows that

$$
\begin{aligned}
&\operatorname{ch}(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))\otimes V(\epsilon_1)) = \operatorname{ch}V((m-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))\\
&+\ \operatorname{ch}V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_{j+1}))+\operatorname{ch}V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_{j-1}))\\
&+\ \operatorname{ch}V((m+1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))+\operatorname{ch}V((m-1)\epsilon_1+(\epsilon_1+\epsilon_2)+(\epsilon_1+\cdots+\epsilon_j)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
T(m,k,j,n) &= \dim\operatorname{Hom}_G(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j)),H^0(G/B,S^k(\mathfrak{u}^*))\otimes H^0(\epsilon_1))\\
&= \dim\operatorname{Hom}_G(V(m\epsilon_1+(\epsilon_1+\cdots+\epsilon_j))\otimes V(\epsilon_1)),H^0(G/B,S^k(\mathfrak{u}^*)))\\
&\geq P(m-1,k,j,n)+P(m,k,j+1,n)+P(m,k,j-1,n),
\end{aligned}
$$

as claimed.

Part (f) follows in similar fashion.

(g) It is well-known that, for $m\geq 2$, $\operatorname{ch}(H^0(m\omega_1))$ is equal to the difference of the mth and the $(m-2)$nd symmetric power of the natural representation. The natural representation has one-dimensional weight spaces and includes the zero weight space. One concludes that the dimension of the zero weight space of the $2l$th symmetric power equals the dimension of the zero weight space of the $(2l+1)$st symmetric power. The same is true for the pair $H^0(2l\omega_1)$ and $H^0((2l+1)\omega_1)$. It follows from Kostant's Theorem [Hum, 24.2] that

$$
\sum_{k\geq 0}P(2l-1,k,1,n)=\sum_{k\geq 0}P(2l,k,1,n). \tag{6.2.14}
$$

From (6.2.10) and (6.2.13) one obtains

$$0 \to H^0(G/B, S^{k-n}(\mathfrak{u}^*) \otimes \epsilon_1) \to H^0(G/B, S^k(\mathfrak{u}^*)) \to H^0(G/B, S^k(\mathfrak{u}^*_{\alpha_n})) \to 0.$$

All three modules have good filtrations. Moreover, by part (i) $\dim \operatorname{Hom}_G(V(2l\epsilon_1 + \epsilon_1), H^0(G/B, S^{k-n}(\mathfrak{u}^*) \otimes \epsilon_1)) = P(2l-1, k-n, 1, n)$. Hence for $l \geq 0$

$$P(2l, k, 1, n) = P(2l-1, k-n, 1, n) + \dim \operatorname{Hom}_G(V(2l\epsilon_1 + \epsilon_1), H^0(G/B, S^k(\mathfrak{u}^*_{\alpha_n}))).$$

Summing over all $k \geq 0$ yields

$$\sum_{k\geq 0} P(2l-1, k, 1, n) = \sum_{k\geq 0} P(2l, k, 1, n) + \sum_{k\geq 0} \dim \operatorname{Hom}_G(V(2l\epsilon_1 + \epsilon_1)), H^0(G/B, S^k(\mathfrak{u}^*_{\alpha_n}))).$$

Comparing with (6.2.14) yields $\dim \operatorname{Hom}_G(V(2l\epsilon_1+\epsilon_1), H^0(G/B, S^k(\mathfrak{u}^*_{\alpha_n}))) = 0$, which forces $P(2l, k, 1, n) = P(2l-1, k-n, 1, n)$, for all $k \geq 0$.

(h) Following [AJ, 3.8], the multiplicity of $\operatorname{ch} V(2l\epsilon_1 + \epsilon_1)$ in $\chi(S^k(\mathfrak{u}^*) \otimes -\epsilon_1)$ equals

$$\sum_{u\in W} (-1)^{\ell(u)} P_k(u \cdot ((2l+1)\epsilon_1) + \epsilon_1).$$

Moreover, by (6.2.11) and (6.2.4),

$$\chi(S^k(\mathfrak{u}^*) \otimes -\epsilon_1) = \operatorname{ch} H^0(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_1) - \operatorname{ch} H^1(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_1)).$$

In addition, from (6.2.8) and (6.2.4), the vanishing of $\operatorname{Hom}_G(V(2l\epsilon_1 + \epsilon_1), H^0(G/B, S^k(\mathfrak{u}^*_{\alpha_n})))$ forces the vanishing of $\operatorname{Hom}_G(V(2l\epsilon_1 + \epsilon_1), H^1(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_1))$, for all k. Hence, $\sum_{u\in W}(-1)^{\ell(u)} P_k(u \cdot ((2l+1)\epsilon_1) + \epsilon_1) = \dim \operatorname{Hom}_G(V(2l\epsilon_1+\epsilon_1), H^0(G/B, S^k(\mathfrak{u}^*) \otimes -\epsilon_1))$, which equals $P(2l, k-n, 1, n)$ by (6.2.4) and (6.2.7). □

Proposition 6.2.3. *Suppose Φ is of type B_n with $n \geq 1$ and $p > 2n$. For $m \geq 0$, $k \geq 0$, and $1 \leq j \leq n$, define*

$$t(m, j, n) = \begin{cases} m + \frac{j+1}{2} & \text{if } j \text{ is odd and } m \text{ is odd,} \\ m + \frac{j}{2} & \text{if } j \text{ is even and } m \text{ is even,} \\ m + 1 + \frac{2n-j}{2} & \text{if } j \text{ is even and } m \text{ is odd,} \\ m + 1 + \frac{2n-j-1}{2} & \text{if } j \text{ is odd and } m \text{ is even.} \end{cases} \tag{6.2.15}$$

Then $P(m, k, j, n) = 0$ whenever $k < t(m, j, n)$.

Proof. By Lemma 6.2.2(c), $P(m, k, j, n) = 0$ if $k < m+1$. We will prove the slightly stronger statement in the proposition inductively. To do so, we make some general observations. Define

$$T'(m, k, j, n) = \sum_{i=2}^{2n} P(m-1, k-i+1, j, n) + \sum_{i=1}^{2n} P(m, k-i+1, j-1, n-1) + P(m, k-n, j, n).$$

Observe that by Lemma 6.2.2(d), $T'(m, k, j, n) = T(m, k, j, n)$. Note further that if r is the smallest value of k for which $T'(m, k, j, n) \neq 0$, then $P(m-1, r-1, j, n) + P(m, r, j-1, n-1) + P(m, r-n, j, n) \neq 0$.

Suppose that $P(m-1, k, j, n) = 0$ whenever $k < t(m-1, j, n)$ and that $P(m, k, j-1, n-1) = 0$ whenever $k < t(m, j-1, n-1)$, then one could conclude that

$$T'(m, k, j, n) = 0 \text{ whenever } k < \min\{t(m-1, j, n) + 1, t(m, j-1, n-1), m+1+n\}. \tag{6.2.16}$$

Moreover, parts (d) and (e) of Lemma 6.2.2 would imply that, for $2 \leq j \leq n-1$,

$$P(m, k, j+1, n) + P(m, k, j-1, n) = 0 \text{ whenever } T'(m, k, j, n) = 0, \tag{6.2.17}$$

and from Lemma 6.2.2(f)

$$P(m, k, n, n) + P(m, k, n-1, n) = 0 \text{ whenever } T'(m, k, n, n) = 0. \tag{6.2.18}$$

In order to prove the proposition, we will use induction on n and or j. If $n = 1$ the claim follows from part (c) of Lemma 6.2.2. Moreover, parts (c) and (d) of the Lemma 6.2.2 imply that the claim holds for $j = 1$ and $n \geq 1$.

Step 1: Here we will show that $P(m, k, j, n) = 0$, whenever $k < t(m, j, n)$ and $m + j$ is odd. We will use induction on j.

Assumption: $P(m,k,l,n)=0$, whenever $k<t(m,l,n)$, $m+l$ is odd, and $l\le j$.

Suppose that $m+j+1$ is odd. Then $m+j-1$ is also odd and the induction assumption implies that (6.2.16) holds. Together with (6.2.17) one obtains

$$P(m,k,j+1,n)=0 \text{ whenever } k<\min\{t(m-1,j,n)+1,t(m,j-1,n-1),m+1+n\}.$$

It suffices therefore to verify that

$$t(m,j+1,n)\le\min\{t(m-1,j,n)+1,t(m,j-1,n-1),m+1+n\}. \tag{6.2.19}$$

From (6.2.15) it follows that

$$t(m,j+1,n)=\begin{cases} m+1+\frac{2n-(j+1)-1}{2}=m+n-\frac{j}{2} & \text{if } j \text{ is even and } m \text{ is even,}\\ m+1+\frac{2n-(j+1)}{2}=m+n-\frac{j-1}{2} & \text{if } j \text{ is odd and } m \text{ is odd,}\end{cases}$$

while

$$t(m-1,j,n)+1=\begin{cases} m+\frac{2n-j}{2}+1=m+n-\frac{j}{2}+1 & \text{if } j \text{ is even and } m \text{ is even,}\\ m+\frac{2n-j-1}{2}+1=m+n-\frac{j-1}{2} & \text{if } j \text{ is odd and } m \text{ is odd,}\end{cases}$$

and

$$t(m,j-1,n-1)=\begin{cases} m+1+\frac{2n-2-(j-1)-1}{2}=m+n-\frac{j}{2} & \text{if } j \text{ is even and } m \text{ is even,}\\ m+1+\frac{2n-2-(j-1)}{2}=m+n-\frac{j-1}{2} & \text{if } j \text{ is odd and } m \text{ is odd.}\end{cases}$$

Inequality (6.2.19) indeed holds and Step 1 is complete.

Step 2: Here we will show that $P(m,k,n,n)=0$, whenever $k<t(m,n,n)$ and $m+n$ is even.

Suppose that $m+n$ is even. Step 1 implies that (6.2.16) holds for $j=n$. Together with (6.2.18) one obtains

$P(m,k,n,n)=0$ whenever $k<\min\{t(m-1,n,n)+1,t(m,n-1,n-1),m+1+n\}$.

It suffices therefore to verify that

$$t(m,n,n)\le\min\{t(m-1,n,n)+1,t(m,n-1,n-1),m+1+n\}.$$

This can easily be done by looking at (6.2.15). It is left to the interested reader.

Step 3: Here we will show that $P(m,k,j,n)=0$ whenever $k<t(m,j,n)$ and $m+j$ is even. We use induction on n and on j. For j we work in decreasing order. The case $j=n$ was settled above.

Assumption: We assume that $P(m,k,l,n-1)=0$ whenever $k<t(m,l,n-1)$. In addition, we assume that $P(m,k,l,n)=0$ whenever $k<t(m,l,n)$, $m+l$ is even, and $l\ge j$.

Suppose that $m+j-1$ is even. The induction assumptions imply that (6.2.16) holds. By (6.2.17) one obtains

$$P(m,k,j-1,n)=0 \text{ whenever } k<\min\{t(m-1,j,n)+1,t(m,j-1,n-1),m+1+n\}.$$

It suffices therefore to verify that

$$t(m,j-1,n)\le\min\{t(m-1,j,n)+1,t(m,j-1,n-1),m+1+n\}. \tag{6.2.20}$$

From (6.2.15) one obtains:

$$t(m,j-1,n)=\begin{cases} m+\frac{j-1}{2} & \text{if } j \text{ is odd and } m \text{ is even,}\\ m+\frac{j}{2} & \text{if } j \text{ is even and } m \text{ is odd.}\end{cases}$$

while

$$t(m-1,j,n)+1=\begin{cases} m+\frac{j-1}{2}+1 & \text{if } j \text{ is odd and } m \text{ is even,}\\ m+\frac{j}{2} & \text{if } j \text{ is even and } m \text{ is odd,}\end{cases}$$

and

$$t(m,j-1,n-1)=\begin{cases} m+\frac{j-1}{2} & \text{if } j \text{ is odd and } m \text{ is even,}\\ m+\frac{j}{2} & \text{if } j \text{ is even and } m \text{ is odd.}\end{cases}$$

This proves inequality (6.2.20). □

THEOREM 6.2.4. *Suppose Φ is of type B_n with $n\ge 2$. Assume that $p>2n$. Let $\lambda=p\omega_1+w\cdot 0$ be a dominant weight. Then*

(a) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda)^{(1)}) = 0$ *for* $0 < i < 2p-2$, *whenever* $\ell(w)$ *is even;*
(b) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda)^{(1)}) = 0$ *for* $0 < i < 2p-3$, *whenever* $\ell(w)$ *is odd;*
(c) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

PROOF. The set of dominant weights of the form $\lambda = p\omega_1 + w\cdot 0$, written in the ϵ-basis, are

$$(p-\ell(w)-1)\epsilon_1 + (\epsilon_1+\cdots+\epsilon_{\ell(w)+1}), \text{ with } 0 \le \ell(w) \le n-1,$$

and

$$(p-\ell(w)-1)\epsilon_1 + (\epsilon_1+\cdots+\epsilon_{2n-\ell(w)}), \text{ with } n \le \ell(w) \le 2n-1.$$

Using Proposition 6.2.3 and Lemma 6.2.2(a) , a direct computation shows that

$$\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot((p-\ell(w)-1)\epsilon_1+(\epsilon_1+\cdots+\epsilon_{\ell(w)+1}))-\epsilon_1) = 0 \text{ whenever } k<t,$$

where

$$t = \begin{cases}(p-1)-\frac{\ell(w)}{2} & \text{for } 0\le \ell(w)\le n-1 \text{ and } \ell(w) \text{ even,}\\ (p-1)-\frac{\ell(w)+1}{2} & \text{for } 0\le \ell(w)\le n-1 \text{ and } \ell(w) \text{ odd,}\end{cases}$$

and

$$\sum_{u\in W}(-1)^{\ell(u)}P_k(u\cdot((p-\ell(w)-1)\epsilon_1+(\epsilon_1+....+\epsilon_{2n-\ell(w)}))-\epsilon_1) = 0 \text{ whenever } k<t,$$

where

$$t = \begin{cases}(p-1)-\frac{\ell(w)}{2} & \text{for } n\le \ell(w)\le 2n-1 \text{ and } \ell(w) \text{ even,}\\ (p-1)-\frac{\ell(w)+1}{2} & \text{for } n\le \ell(w)\le 2n-1 \text{ and } \ell(w) \text{ odd.}\end{cases}$$

Parts (a) and (b) follow from Proposition 2.7.1. Note that $i = 2k+\ell(w)$.

Let λ be the lowest dominant weight of the form $p\omega_1 + w\cdot 0$. Then $\lambda = (p-2n+1)\epsilon_1$ and $\ell(w) = 2n-1$. We will show that

$$\sum_{u\in W}(-1)^{\ell(u)}P_{p-n-1}(u\cdot((p-2n+1)\epsilon_1)-\epsilon_1) \neq 0. \tag{6.2.21}$$

By Lemma 6.2.2(a) this is equivalent to showing that $P(p-2n-1, p-n-1, 1, n)$ is not zero. Lemma 6.2.2(b) and (h) imply that

$$P(2l+1,k,1,n) = P(2l,k-n,1,n) + P(2l+1,k,1,n-1). \tag{6.2.22}$$

Note that (6.2.22) also holds for $l=-1$. Obviously $P(2l-1,2l,1,1) = 1$. It follows inductively from (6.2.22) that $P(2l-1,2l,1,n)\neq 0$, for all $n\ge 0$, $l\ge 0$. From Lemma 6.2.2(g) one obtains now that $P(2l,2l+n,1,n)\neq 0$. Setting $2l = p-2n-1$ yields $P(p-2n-1,p-n-1,1,n)\neq 0$. Hence, (6.2.21) holds. In Proposition 2.7.1, $i = 2k-\ell(w) = 2(p-n-1)+2n-1 = 2p-3$ and one obtains $\mathrm{H}^{2p-3}(G, H^0(\lambda)\otimes H^0(\lambda)^{(1)})\neq 0$. The weight λ is the lowest non-zero weight in its linkage class. Part (c) of the theorem follows now from the discussion after Theorem 2.5.1. □

6.3. Type B_3. Let Φ be of type B_3 with $p > h = 6$ (so $p\ge 7$). From the discussion in Section 6.1 and Theorem 6.2.4, in order to have $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})\neq 0$ for $0<i<2p-3$, we must have $\lambda = p\omega_3 + w\cdot 0$ for $w\in W$. With the aid of MAGMA [BC, BCP] or other software, one can explicitly compute all $w\cdot 0$ and determine which resulting λ are dominant. Further, $\lambda-\omega_3$ must be a weight of $S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*)$. By direct computation, one can determine the least possible value of k for which $\lambda-\omega_3$ can be a weight of $S^k(\mathfrak{u}^*)$. The following table summarizes the weights which can give a value of $i<2p-6$.

$\lambda = p\omega_3 + w\cdot 0$	$\ell(w)$	k	$i = 2k+\ell(w)$
$(p-6)\omega_3 + 2\omega_2$	3	$p-5$	$2p-7$
$(p-6)\omega_3 + \omega_1$	5	$p-6$	$2p-7$
$(p-6)\omega_3$	6	$p-7$	$2p-8$

LEMMA 6.3.1. *Suppose that Φ is of type B_3 with $p\ge 7$. Let $\lambda = p\mu + w\cdot 0\in X(T)_+$ with $\mu\in X(T)_+$ and $w\in W$.*
(a) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0<i<2p-8$.
(b) *If* $\mathrm{H}^{2p-8}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})\neq 0$, *then* $\lambda = (p-6)\omega_3$.

(c) $\mathrm{H}^{2p-8}(G, H^0((p-6)\omega_3) \otimes H^0((p-6)\omega_3^*)^{(1)}) = k$.
(d) *If* $\mathrm{H}^{2p-7}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$, *then* $\lambda = (p-6)\omega_3 + \omega_1$ *or* $\lambda = (p-6)\omega_3 + 2\omega_2$.
(e) $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) = k$.

Proof. Parts (a), (b), and (d) follow from the discussion preceding the lemma. Part (c) follows from Proposition 2.7.1 and Lemma 6.3.3 below with $m = p-7$. Since the weights in part (d) are larger than $(p-6)\omega_3$, by Theorem 2.5.1 and Theorem 6.2.4, we obtain part (e). □

Remark 6.3.2. The weights in part (d) also appear to give cohomology classes as verified for $p = 7, 11, 13$ by computer. For $p = 7$, $\lambda = (p-6)\omega_3 + \omega_1$ gives a one-dimensional cohomology group. But for $p = 11, 13$, one gets a two-dimensional cohomology group. For all three primes, the weight $\lambda = (p-6)\omega_3 + 2\omega_2$ gives a one-dimensional cohomology group.

Lemma 6.3.3. *Suppose that Φ is of type B_3. Let $m \geq 0$ be an even integer. Then*

$$\sum_{u \in W} (-1)^{\ell(u)} P_m(u \cdot ((m+1)\omega_3) - \omega_3) = 1.$$

Proof. Let n be such that $m = 2n$. For $n = 0$, the claim readily follows, so we assume that $n \geq 1$.[1] We work with the epsilon basis for the root system. Then the positive roots are ϵ_1, ϵ_2, ϵ_3, $\epsilon_1+\epsilon_2$, $\epsilon_1+\epsilon_3$, $\epsilon_2+\epsilon_3$, $\epsilon_1-\epsilon_2$, $\epsilon_1-\epsilon_3$, and $\epsilon_2-\epsilon_3$. Further $\omega_3 = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3)$. Relative to the ϵ basis, for any $u \in W$, $u(\epsilon_i) = \pm\epsilon_j$. That is, u permutes the ϵ_i up to a sign.

For $u \in W$, set $x_u := u \cdot ((m+1)\omega_3) - \omega_3$. Using the fact that $2\rho = 5\epsilon_1 + 3\epsilon_2 + \epsilon_3$, one finds that

$$x_u = u((n+3)\epsilon_1 + (n+2)\epsilon_2 + (n+1)\epsilon_3) - 3\epsilon_1 - 2\epsilon_2 - \epsilon_1. \tag{6.3.1}$$

By direct calculation, one can identify all $u \in W$ for which x_u is a positive root. There are twelve such elements which are summarized in the following table (using permutation notation) along with the parity of their lengths. An element marked with a superscript negative sign denotes the operation which consists of the given permutation of the ϵ_is followed by sending ϵ_3 to $-\epsilon_3$. For example, let $u = (123)^-$. Then $u(\epsilon_1) = \epsilon_2$, $u(\epsilon_2) = -\epsilon_3$, and $u(\epsilon_3) = \epsilon_1$.

u	(1)	(12)	(13)	(23)	(123)	(132)	$(1)^-$	$(12)^-$	$(13)^-$	$(23)^-$	$(123)^-$	$(132)^-$
$\ell(u)$	even	odd	odd	odd	even	even	odd	even	even	even	odd	odd

For these twelve u, using (6.3.1), one can explicitly compute x_u. The values are summarized in the following table. Recall that $m = 2n$. For small values of n, some of these cannot be sums of positive roots. The necessary condition on n is given in the third column.

u	$x_u := u \cdot ((m+1)\omega_3) - \omega_3$	positive root sum
(1)	$n\epsilon_1 + n\epsilon_2 + n\epsilon_3$	$n \geq 1$
(12)	$(n-1)\epsilon_1 + (n+1)\epsilon_2 + n\epsilon_3$	$n \geq 1$
(13)	$(n-2)\epsilon_1 + n\epsilon_2 + (n+2)\epsilon_3$	$n \geq 2$
(23)	$n\epsilon_1 + (n-1)\epsilon_2 + (n+1)\epsilon_3$	$n \geq 1$
(123)	$(n-2)\epsilon_1 + (n+1)\epsilon_2 + (n+1)\epsilon_3$	$n \geq 2$
(132)	$(n-1)\epsilon_1 + (n-1)\epsilon_2 + (n+2)\epsilon_3$	$n \geq 1$
$(1)^-$	$n\epsilon_1 + n\epsilon_2 - (n+2)\epsilon_3$	$n \geq 2$
$(12)^-$	$(n-1)\epsilon_1 + (n+1)\epsilon_2 - (n+2)\epsilon_3$	$n \geq 2$
$(13)^-$	$(n-2)\epsilon_1 + n\epsilon_2 - (n+4)\epsilon_3$	$n \geq 6$
$(23)^-$	$n\epsilon_1 + (n-1)\epsilon_2 - (n+3)\epsilon_3$	$n \geq 4$
$(123)^-$	$(n-2)\epsilon_1 + (n+1)\epsilon_2 - (n+3)\epsilon_3$	$n \geq 4$
$(132)^-$	$(n-1)\epsilon_1 + (n-1)\epsilon_2 - (n+4)\epsilon_3$	$n \geq 6$

We need to compute the appropriate alternating sum of the values of $P_{2n}(x_u)$ for these twelve values of u. We show below that there are four pairs of us for which the lengths have opposite parity and the values of $P_{2n}(x_u)$ are the same. Hence those cancel each other out. We will further show that there is also

[1] Indeed, for small values of n the claim can be verified by hand, and it has been verified for $n \leq 6$ using MAGMA.

a relationship between the remaining partition functions that will lead to the desired claim. To see these relationships, we make a few observations whose proofs are left to the interested reader.

Observation 6.3.4. Let $x = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$ with $a_1 + a_2 + a_3 = 3n$. Suppose that x is expressed as a sum of $2n$ positive roots.

(a) At least n of the roots have the form $\epsilon_i + \epsilon_j$ (not necessarily all the same).
(b) For any pair $i, j \in \{1, 2, 3\}$ (with $i \neq j$), if $a_i + a_j = 2n + c$, then the root sum decomposition contains at least c copies of $\epsilon_i + \epsilon_j$.

Observation 6.3.5. Let $x = a_1\epsilon_1 + a_2\epsilon_2 - a_3\epsilon_3$ with $1 \leq a_1, a_2 < a_3$ and $a_1 + a_3 > 2n$. Suppose that x is expressed as a sum of $2n$ positive roots. Then at least one of the roots is $\epsilon_1 - \epsilon_3$ and at least one is $\epsilon_2 - \epsilon_3$.

We now identify the four pairs (of opposite parity) where $P_{2n}(x_u)$ is the same.

Case 1. (13) and (132).
If $n = 1$, as noted above, $x_{(13)} = -\epsilon_1 + \epsilon_2 + 3\epsilon_3$ cannot be expressed as a sum of positive roots. On the other hand, $x_{(123)} = 3\epsilon_3$ can be. However, it cannot be expressed as a sum of $2n = 2$ positive roots. So we assume $n \geq 2$. If $x_{(13)}$ is expressed as a sum of $2n$ positive roots, by Observation 6.3.4(b), at least one of the roots is $\epsilon_2 + \epsilon_3$ (in fact, at least two). Hence, $P_{2n}(x_{(13)}) = P_{2n-1}(x_{(13)} - (\epsilon_2 + \epsilon_3)) = P_{2n-1}((n-2)\epsilon_1 + (n-1)\epsilon_2 + (n+1)\epsilon_3)$. Similarly, if $x_{(132)}$ is expressed as a sum of $2n$ positive roots, at least one of the roots is $\epsilon_1 + \epsilon_3$ (and one is $\epsilon_2 + \epsilon_3$). Hence, $P_{2n}(x_{(132)}) = P_{2n-1}((n-2)\epsilon_1 + (n-1)\epsilon_2 + (n+1)\epsilon_3) = P_{2n}(x_{(13)})$.

For the remaining three pairs, if n is not sufficiently large for x_u to admit a positive root sum decomposition, then $P_{2n}(x_u) = 0$ in both cases. So we assume in what follows that n is sufficiently large to admit a root sum decomposition.

Case 2. $(1)^-$ and $(12)^-$.
If $x_{(1)^-}$ is expressed as a sum of $2n$ positive roots, by Observation 6.3.5, at least one of the roots is $\epsilon_1 - \epsilon_3$. Hence, removing this root, $P_{2n}(x_{(1)^-}) = P_{2n-1}((n-1)\epsilon_1 + n\epsilon_2 - (n+1)\epsilon_3)$. Similarly, again by Observation 6.3.5, if $x_{(12)^-}$ is expressed as a sum of $2n$ positive roots, then at least one of the roots is $\epsilon_2 - \epsilon_3$. Hence, $P_{2n}(x_{(12)^-}) = P_{2n-1}((n-1)\epsilon_1 + n\epsilon_2 - (n+1)\epsilon_3) = P_{2n}(x_{(1)^-})$.

Case 3. $(13)^-$ and $(132)^-$.
Similar to Case 2, by removing an $\epsilon_2 - \epsilon_3$ for $(13)^-$ and removing an $\epsilon_1 - \epsilon_3$ for $(132)^-$, one finds that $P_{2n}(x_{(12)^-}) = P_{2n-1}((n-2)\epsilon_1 + (n-1)\epsilon_2 - (n+3)\epsilon_3) = P_{2n}(x_{(132)^-})$.

Case 4. $(23)^-$ and $(123)^-$.
Again, similar to Case 2 with a slight generalization of Observation 6.3.5, by removing two copies of $\epsilon_1 - \epsilon_3$ for $(23)^-$ and two copies of $\epsilon_2 - \epsilon_3$ for $(123)^-$, one finds that $P_{2n}(x_{(23)^-}) = P_{2n-2}((n-2)\epsilon_1 + (n-1)\epsilon_2 - (n+1)\epsilon_3) = P_{2n}(x_{(123)^-})$.

From Cases 1-4, we have that

$$\sum_{u \in W} (-1)^{\ell(u)} P_m(u \cdot ((m+1)\omega_3) - \omega_3) = P_{2n}(x_{(1)}) - P_{2n}(x_{(12)}) - P_{2n}(x_{(23)}) + P_{2n}(x_{(123)}). \tag{6.3.2}$$

We now deduce several relationships among the terms on the right hand side. If $n = 1$, only the first three terms can be non-zero, and one can readily check that the claim holds. So we assume that $n \geq 2$. Note that the following argument does still hold even when $n = 1$.

Consider the identity element (1). Write $P_{2n}(x_{(1)}) = M_1 + M_2 + M_3$ where M_1 denotes the number of root sum decompositions which contain an $\epsilon_1 + \epsilon_2$, M_2 denotes the number which contain an $\epsilon_1 + \epsilon_3$ but not an $\epsilon_1 + \epsilon_2$, and M_3 denotes the number which contain neither an $\epsilon_1 + \epsilon_2$ nor an $\epsilon_1 + \epsilon_3$. For M_1, by assumption, the decomposition contains an $\epsilon_1 + \epsilon_2$. Removing this root gives

$$M_1 = P_{2n-1}((n-1)\epsilon_1 + (n-1)\epsilon_2 + n\epsilon_3). \tag{6.3.3}$$

For M_2, by assumption, the decomposition contains an $\epsilon_1 + \epsilon_3$. Removing this root gives

$$M_2 = P^*_{2n-1}((n-1)\epsilon_1 + n\epsilon_2 + (n-1)\epsilon_3), \tag{6.3.4}$$

where P^* denotes the fact that we are only counting decompositions which contain no copies of $\epsilon_1 + \epsilon_2$. By assumption, M_3 is the number of root decompositions of $n\epsilon_1 + n\epsilon_2 + n\epsilon_3$ (into $2n$ positive roots) which do not contain an $\epsilon_1 + \epsilon_2$ nor an $\epsilon_1 + \epsilon_3$. By Observation 6.3.4(a), any such decomposition contains at least n

copies of $\epsilon_2 + \epsilon_3$. Removing those leaves $n\epsilon_1$ which must be expressed as a sum of n positive roots without using $\epsilon_1 + \epsilon_2$ nor $\epsilon_1 + \epsilon_3$. There is clearly only one such decomposition (using n copies of ϵ_1). Hence, $M_3 = 1$.

Consider now the word (12). Write $P_{2n}(x_{(12)}) = N_1 + N_2$ where N_1 denotes the number of root sum decompositions which contain at least one copy of $\epsilon_1 + \epsilon_2$ and N_2 denotes the number which do not contain an $\epsilon_1 + \epsilon_2$. In the first case, by removing an $\epsilon_1 + \epsilon_2$, we have

$$N_1 = P_{2n-1}((n-2)\epsilon_1 + n\epsilon_2 + n\epsilon_3). \tag{6.3.5}$$

In the second case (as well as the first case), by Observation 6.3.4(b), any decomposition must include an $\epsilon_2 + \epsilon_3$. Removing that, we see that

$$N_2 = P^*_{2n-1}((n-1)\epsilon_1 + n\epsilon_2 + (n-1)\epsilon_3) = M_2$$

from (6.3.4).

From Observation 6.3.4(b), by removing an $\epsilon_1 + \epsilon_3$,

$$P_{2n}(x_{(23)}) = P_{2n-1}((n-1)\epsilon_1 + (n-1)\epsilon_2 + n\epsilon_3) = M_1,$$

where the second equality follows from (6.3.3). From Observation 6.3.4(b), by removing an $\epsilon_2 + \epsilon_3$,

$$P_{2n}(x_{(123)}) = P_{2n-1}((n-2)\epsilon_1 + n\epsilon_2 + n\epsilon_3) = N_1,$$

where the second equality follows from (6.3.5)[2].

From (6.3.2) and the preceding relationships, we have

$$\begin{aligned}\sum_{u\in W}(-1)^{\ell(u)}P_m(u\cdot((m+1)\omega_3)-\omega_3) &= P_{2n}(x_{(1)}) - P_{2n}(x_{(12)}) - P_{2n}(x_{(23)}) + P_{2n}(x_{(123)})\\ &= M_1 + M_2 + M_3 - N_1 - N_2 - P_{2n}(x_{(23)}) + P_{2n}(x_{(123)})\\ &= M_1 + M_2 + 1 - N_1 - M_2 - M_1 + N_1\\ &= 1\end{aligned}$$

as claimed.

□

6.4. Type B_4. Let Φ be of type B_4 with $p > h = 8$ (so $p \geq 11$). As discussed in Section 6.3 for type B_3, in order to have $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $0 < i < 2p-3$, we must have $\lambda = p\omega_4 + w\cdot 0$ for $w \in W$. Again, by direct computation with MAGMA, the following table summarizes the weights which can give a value of $i < 2p - 6$.

$\lambda = p\omega_4 + w\cdot 0$	$\ell(w)$	k	$i = 2k + \ell(w)$
$(p-8)\omega_4 + 2\omega_2$	$p-7$	7	$2p-7$
$(p-8)\omega_4 + \omega_1$	$p-8$	9	$2p-7$
$(p-8)\omega_4$	$p-9$	10	$2p-8$

Lemma 6.4.1. *Suppose that Φ is of type B_4 with $p \geq 11$. Let $\lambda = p\mu + w\cdot 0 \in X(T)_+$ with $\mu \in X(T)_+$ and $w \in W$.*

(a) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-8$.
(b) *If* $\mathrm{H}^{2p-8}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$, *then* $\lambda = (p-8)\omega_4$.
(c) $\mathrm{H}^{2p-8}(G, H^0((p-8)\omega_4) \otimes H^0((p-8)\omega_4^*)^{(1)}) = k$.
(d) *If* $\mathrm{H}^{2p-7}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$, *then* $\lambda = (p-8)\omega_4 + \omega_1$ *or* $\lambda = (p-8)\omega_4 + 2\omega_2$.
(e) $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) = k$.

Proof. Parts (a), (b), and (d) follow from the discussion preceding the lemma. Part (c) follows from Proposition 2.7.1 and Lemma 6.4.2 below with $m = p-9$. Since the weights in part (d) are larger than $(p-8)\omega_4$, by Theorem 2.5.1 and Theorem 6.2.4, we obtain part (e). □

[2]If $n = 1$, $x_{(123)}$ cannot be expressed as a sum of positive roots. However, in that case, N_1 is necessarily zero, and so we still have $P_{2n}(x_{(123)}) = N_1$.

Lemma 6.4.2. *Suppose that Φ is of type B_4. Let $m \geq 0$ be an even integer. Then*

$$\sum_{u\in W}(-1)^{\ell(u)}P_m(u\cdot((m+1)\omega_4)-\omega_4)=1.$$

Proof. The arguments to follow are quite similar to those in the proof of Lemma 6.3.3. Let n be such that $m = 2n$. For $n = 0$, the claim readily follows, so we assume that $n \geq 1$. As with the proof of Lemma 6.3.3, we work with the epsilon basis for the root system. Then the positive roots are ϵ_1, ϵ_2, ϵ_3, ϵ_4, $\epsilon_1+\epsilon_2$, $\epsilon_1+\epsilon_3$, $\epsilon_1+\epsilon_4$, $\epsilon_2+\epsilon_3$, $\epsilon_2+\epsilon_4$, $\epsilon_3+\epsilon_4$, $\epsilon_1-\epsilon_2$, $\epsilon_1-\epsilon_3$, $\epsilon_1-\epsilon_4$, $\epsilon_2-\epsilon_3$, $\epsilon_2-\epsilon_4$, and $\epsilon_3-\epsilon_4$. Further $\omega_4=\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$. Relative to the ϵ basis, for any $u\in W$, $u(\epsilon_i)=\pm\epsilon_j$. That is, u permutes the ϵ_i up to a sign.

For $u \in W$, let $x_u := u\cdot((m+1)\omega_4)-\omega_4$. Using the fact that $2\rho = 7\epsilon_1+5\epsilon_2+3\epsilon_3+\epsilon_4$, one finds that

$$x_u = u((n+4)\epsilon_1+(n+3)\epsilon_2+(n+2)\epsilon_3+(n+1)\epsilon_4)-4\epsilon_1-3\epsilon_2-2\epsilon_3-\epsilon_1. \tag{6.4.1}$$

By direct calculation, one finds that if u sends any ϵ_i to $-\epsilon_j$ (any j), then x_u is either not expressible as a sum of positive roots or requires at least $2n+1$ roots to do so. Therefore, the only u that can contribute to the alternating sum under consideration are those u for which $u(\epsilon_i)=\epsilon_j$. That is, u is simply one of the 24 permutations of the ϵ_is.

Let $u \in S_4 \subset W$. From (6.4.1), one finds that $x_u = a_1\epsilon_1+a_2\epsilon_2+a_3\epsilon_3+a_4\epsilon_4$ where $a_1+a_2+a_3+a_4=4n$. Since the positive roots are of the form ϵ_i, $\epsilon_i+\epsilon_j$, or $\epsilon_i-\epsilon_j$, for this to be expressed as a sum of $2n$ roots, each such root must be of the form $\epsilon_i+\epsilon_j$. That is the other two types of roots are not allowable. Similar to the arguments in the proof of Lemma 6.3.3, one can further see the following.

Observation 6.4.3. Suppose that $a_1\epsilon_1+a_2\epsilon_2+a_3\epsilon_3+a_4\epsilon_4$ is expressed as a sum of $2n$ positive roots where $a_1+a_2+a_3+a_4=4n$. For any pair $i,j\in\{1,2,3,4\}$ (with $i\neq j$), if $a_i+a_j=2n+c$, then the root sum decomposition contains at least c copies of $\epsilon_i+\epsilon_j$.

Using Observation 6.4.3, by direct calculation, one can show that the 18 permutations u for which $u(\epsilon_1)\neq\epsilon_1$ can be separated into 9 pairs of opposite parity having equal values of $P_{2n}(x_u)$. Hence the terms for those values of u cancel in the alternating sum. For example, consider the permutations (12) and (12)(43) of opposite parity. From (6.4.1), $x_{(12)}=(n-1)\epsilon_1+(n+1)\epsilon_2+n\epsilon_3+n\epsilon_4$. By Observation 6.4.3, a decomposition of $x_{(12)}$ must contain at least one copy of $\epsilon_2+\epsilon_3$ (as well as a copy of $\epsilon_2+\epsilon_4$). Subtracting that root shows that

$$P_{2n}(x_{(12)})=P_{2n-1}((n-1)\epsilon_1+n\epsilon_2+(n-1)\epsilon_3+n\epsilon_4).$$

On the other hand, $x_{(12)(34)}=(n-1)\epsilon_1+(n+1)\epsilon_2+(n-1)\epsilon_3+(n+1)\epsilon_4$. Here, $x_{(12)(34)}$ must contain a copy of $\epsilon_2+\epsilon_4$ (in fact, at least two copies). Subtracting this root gives

$$P_{2n}(x_{(12)(34)})=P_{2n-1}((n-1)\epsilon_1+n\epsilon_2+(n-1)\epsilon_3+n\epsilon_4)=P_{2n}(x_{(12)}).$$

The eight other pairings (which may not be unique) are (13) with (13)(24); (14) with (14)(23); (123) with (1243); (132) with (1342); (124) with (1234); (142) with (1432); (134) with (1324); and (143) with (1423). We leave the details to the interested reader.

That leaves the six values of u for which $u(\epsilon_1)=\epsilon_1$: (1), (23), (24), (34), (234), and (243). However, as above, one can show that $P_{2n}(x_{(24)})=P_{2n}(x_{(243)})$. So those terms cancel as well and we are reduced to

$$\sum_{u\in W}(-1)^{\ell(u)}P_m(u\cdot((m+1)\omega_4)-\omega_4)=P_{2n}(x_{(1)})-P_{2n}(x_{(23)})-P_{2n}(x_{(34)})+P_{2n}(x_{(234)}).$$

From (6.4.1), we have $x_{(1)}=n\epsilon_1+n\epsilon_2+n\epsilon_3+n\epsilon_4$. Write $P_{2n}(x_{(1)})=M_1+M_2+M_3$ where M_1 denotes the number of root sum decompositions which contain at least one copy of $\epsilon_2+\epsilon_3$, M_2 denotes the number which contain no copies of $\epsilon_2+\epsilon_3$ but contain at least one copy of $\epsilon_1+\epsilon_2$, and M_3 denotes the number which contain neither an $\epsilon_2+\epsilon_3$ nor an $\epsilon_1+\epsilon_2$. By assumption, subtracting a copy of $\epsilon_2+\epsilon_3$, we have

$$M_1=P_{2n-1}(n\epsilon_1+(n-1)\epsilon_2+(n-1)\epsilon_3+n\epsilon_4). \tag{6.4.2}$$

For M_2, subtracting a copy of $\epsilon_1+\epsilon_2$ gives

$$M_2=P^*_{2n-1}((n-1)\epsilon_1+(n-1)\epsilon_2+n\epsilon_3+n\epsilon_4), \tag{6.4.3}$$

where the P^* denotes the fact that the sum is only over those decompositions which do not contain a copy of $\epsilon_2+\epsilon_3$. For M_3, in order to get the $n\epsilon_2$ appearing in $x_{(1)}$, there must be exactly n copies of $\epsilon_2+\epsilon_4$. But then the remaining n factors must all be $\epsilon_1+\epsilon_3$. In other words, $M_3=1$.

From (6.4.1), we have $x_{(23)} = n\epsilon_1 + (n-1)\epsilon_2 + (n+1)\epsilon_3 + n\epsilon_4$. Write $P_{2n}(x_{(23)}) = N_1 + N_2$ where N_1 denotes the number of root sum decompositions which contain at least one copy of $\epsilon_2+\epsilon_3$ and N_2 denotes the number which contain no copies of $\epsilon_2+\epsilon_3$. Subtracting a copy of $\epsilon_2+\epsilon_3$, we have

$$N_1 = P_{2n-1}(n\epsilon_1 + (n-2)\epsilon_2 + n\epsilon_3 + n\epsilon_4). \tag{6.4.4}$$

For N_2, by Observation 6.4.3, any decomposition of $x_{(23)}$ contains at least one copy of $\epsilon_1+\epsilon_3$ (as well as a copy of $\epsilon_3+\epsilon_4$). Subtracting the $\epsilon_1+\epsilon_3$ gives

$$N_2 = P^*_{2n-1}((n-1)\epsilon_1 + (n-1)\epsilon_2 + n\epsilon_3 + n\epsilon_4) = M_2$$

from (6.4.3).

From (6.4.1), we have $x_{(34)} = n\epsilon_1 + n\epsilon_2 + (n-1)\epsilon_3 + (n+1)\epsilon_4$. From Observation 6.4.3, any decomposition of $x_{(34)}$ contains at least one copy of $\epsilon_2+\epsilon_4$ (as well as a copy of $\epsilon_1+\epsilon_4$). Subtracting the $\epsilon_2+\epsilon_4$ gives

$$P_{2n}(x_{(34)}) = P_{2n-1}(n\epsilon_1 + (n-1)\epsilon_2 + (n-1)\epsilon_3 + n\epsilon_4) = M_1$$

from (6.4.2).

From (6.4.1), we have $x_{(234)} = n\epsilon_1 + (n-2)\epsilon_2 + (n+1)\epsilon_3 + (n+1)\epsilon_4$. From Observation 6.4.3, any decomposition of $x_{(234)}$ contains at least one copy of $\epsilon_3+\epsilon_4$ (in fact, at least two copies). Subtracting this gives

$$P_{2n}(x_{(234)}) = P_{2n-1}(n\epsilon_1 + (n-2)\epsilon_2 + n\epsilon_3 + n\epsilon_4) = N_1$$

from (6.4.4).

In summary, we have

$$\begin{aligned}\sum_{u\in W}(-1)^{\ell(u)}P_m(u\cdot((m+1)\omega_4)-\omega_4) &= P_{2n}(x_{(1)}) - P_{2n}(x_{(23)}) - P_{2n}(x_{(34)}) + P_{2n}(x_{(234)})\\ &= M_1 + M_2 + M_3 - N_1 - N_2 - P_{2n}(x_{34}) + P_{2n}(x_{234})\\ &= M_1 + M_2 + 1 - N_1 - M_2 - M_1 + N_1\\ &= 1\end{aligned}$$

as claimed. □

6.5. Type B_5. Let Φ be of type B_5 with $p > h = 10$ (so $p \geq 11$). As discussed in Section 6.3 for type B_3, in order to have $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $0 < i < 2p-3$, we must have $\lambda = p\omega_5 + w\cdot 0$ for $w \in W$. Specifically, substituting $n = 5$ into (6.1.1) gives

$$i \geq 2p - \frac{p-25}{2}. \tag{6.5.1}$$

We obtain the following.

Lemma 6.5.1. *Suppose that Φ is of type B_5 with $p \geq 11$. Let $\lambda = p\omega_5 + w\cdot 0 \in X(T)_+$ with $w \in W$.*

(a) *If $p = 17$ or $p \geq 23$, then $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i \leq 2p-3$.*
(b) *Suppose $p = 11$. Then*
  (i) *$\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i < 2p-7$;*
  (ii) *if $\mathrm{H}^{2p-7}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$, then $\lambda = (p-10)\omega_5 = \omega_5$;*
  (iii) *$\mathrm{H}^{2p-7}(G, H^0(\omega_5)\otimes H^0(\omega_5^*)^{(1)}) = k$;*
  (iv) *if $\mathrm{H}^{2p-6}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$, then $\lambda = \omega_1+\omega_5$ or $\lambda = 2\omega_2+\omega_5$;*
  (v) *$\mathrm{H}^{2p-6}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = k$ for $\lambda = \omega_1+\omega_5$ or $\lambda = 2\omega_2+\omega_5$;*
  (vi) *$\mathrm{H}^{2p-7}(G(\mathbb{F}_p), k) = k$.*
(c) *Suppose $p = 13$. Then*
  (i) *$\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i < 2p-5$;*
  (ii) *if $\mathrm{H}^{2p-5}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$, then $\lambda = (p-10)\omega_5 = 3\omega_5$;*
  (iii) *$\mathrm{H}^{2p-5}(G, H^0(3\omega_5)\otimes H^0(3\omega_5^*)^{(1)}) = k$;*
  (iv) *if $\mathrm{H}^{2p-4}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$, then $\lambda = \omega_1+3\omega_5$ or $\lambda = 2\omega_2+3\omega_5$;*
  (v) *$\dim \mathrm{H}^{2p-4}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 2$ for $\lambda = \omega_1+3\omega_5$;*
  (vi) *$\mathrm{H}^{2p-4}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = k$ for $\lambda = 2\omega_2+3\omega_5$;*
  (vii) *$\mathrm{H}^{2p-5}(G(\mathbb{F}_p), k) = k$.*
(d) *Suppose $p = 19$. Then*

(i) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-3$*;*
(ii) *if* $\mathrm{H}^{2p-3}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$*, then* $\lambda = (p-10)\omega_5 = 9\omega_5$*;*
(iii) $\dim \mathrm{H}^{2p-3}(G, H^0(9\omega_5) \otimes H^0(9\omega_5^*)^{(1)}) = 15$.

Proof. For $p \geq 23$, part (a) follows from (6.5.1). Parts (b)(i)-(v), (c)(i)-(vi), and (d) as well as part (a) for $p = 17$ follow by explicitly computing (with the aid of MAGMA) all possible $w \cdot 0$, and then computing partition functions by hand or with the aid of MAGMA. For $p = 11$, since the weights in part (b)(iv) are larger than that in (b)(ii), part (b)(vi) follows from Theorem 2.5.1 and Theorem 6.2.4. Similarly, part (c)(vii) follows. □

6.6. Type B_6. Let Φ be of type B_6 with $p > h = 12$ (so $p \geq 13$). As discussed in Section 6.3 for type B_3, in order to have $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $0 < i < 2p-3$, we must have $\lambda = p\omega_6 + w \cdot 0$ for $w \in W$. Recall the arguments in Section 6.1. Specifically, substituting $n = 6$ into (6.1.1) gives

$$i \geq 2p + (p-18). \tag{6.6.1}$$

We obtain the following.

Lemma 6.6.1. *Suppose that Φ is of type B_6 with $p \geq 13$. Let $\lambda = p\omega_6 + w \cdot 0 \in X(T)_+$ with $w \in W$.*

(a) *If* $p \geq 17$*, then* $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i \leq 2p-3$.
(b) *Suppose* $p = 13$*. Then*
  (i) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-5$*;*
  (ii) *if* $\mathrm{H}^{2p-5}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$*, then* $\lambda = (p-12)\omega_6 = \omega_6$*;*
  (iii) $\mathrm{H}^{2p-5}(G, H^0(\omega_6) \otimes H^0(\omega_6^*)^{(1)}) = k$*;*
  (iv) *if* $\mathrm{H}^{2p-4}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$*, then* $\lambda = \omega_1 + \omega_6$ *or* $\lambda = 2\omega_2 + \omega_6$*;*
  (v) $\mathrm{H}^{2p-4}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = k$ *for* $\lambda = \omega_1 + \omega_6$ *or* $\lambda = 2\omega_2 + \omega_6$*;*
  (vi) $\mathrm{H}^{2p-5}(G(\mathbb{F}_p), k) = k$.

Proof. Part (a) follows from (6.6.1). Parts (b)(i)-(v) follow by explicitly computing (with the aid of MAGMA) all possible $w \cdot 0$, and then computing partition functions by hand or with the aid of MAGMA. Since the weights in part (b)(iv) are larger than that in part (b)(ii), part (b)(vi) follows from Theorem 2.5.1 and Theorem 6.2.4. □

6.7. Summary for type B.

Theorem 6.7.1. *Suppose Φ is of type B_n with $n \geq 3$. Assume that $p > 2n$.*

(a) *If* $n \geq 7$ *or* $p > 13$ *when* $n \in \{5, 6\}$*, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$*;*
  (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.
(b) *If* $n \in \{5, 6\}$ *and* $p = 13$*, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-5$*;*
  (ii) $\mathrm{H}^{2p-5}(G(\mathbb{F}_p), k) = k$.
(c) *If* $n = 5$ *and* $p = 11$*, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-7$*;*
  (ii) $\mathrm{H}^{2p-7}(G(\mathbb{F}_p), k) = k$.
(d) *If* $n \in \{3, 4\}$*, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-8$*;*
  (ii) $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) = k$.

Proof. This follows from the discussion in Section 6.1, Theorem 6.2.4, Lemma 6.3.1, Lemma 6.4.1, Lemma 6.5.1, and Lemma 6.6.1. □

7. Type G_2

Assume throughout this section that Φ is of type G_2 and that $p > h = 6$ (so $p \geq 7$). Following Section 2, our goal is to find the least $i > 0$ such that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)) \neq 0$ for some $\lambda \in X(T)_+$.

7.1. Restrictions. Suppose that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i > 0$ and $\lambda = p\mu + w \cdot 0$ with $\mu \in X(T)_+$ and $w \in W$. From Proposition 2.8.1(c), $i \geq (p-1)\langle \mu, \tilde{\alpha}^\vee \rangle - 1$. Consider the two fundamental weights ω_1 and ω_2. Note that $\omega_1 = \alpha_0$ and $\omega_2 = \tilde{\alpha}$. Furthermore, we have $\langle \omega_1, \tilde{\alpha}^\vee \rangle = 1$ and $\langle \omega_2, \tilde{\alpha}^\vee \rangle = 2$. Therefore, unless $\mu = \omega_1 = \alpha_0$, we have $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 2$ and $i \geq 2p - 3$.

Suppose now that $\lambda = p\omega_1 + w \cdot 0$ for some $w \in W$. In order to have $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$, as discussed in Section 2.7, λ must be dominant and $\lambda - \omega_1$ must be a weight of $S^j(\mathfrak{u}^*)$ for some j. In other words, $\lambda - \omega_1$ must be expressible as a non-negative linear combination of positive roots. By direct calculation (by hand or with the aid of MAGMA), one can identify all possible λ. These are listed in the following table. As usual, $s_i := s_{\alpha_i}$ and e is the identity element.

w	$\ell(w)$	$\lambda = p\omega_1 + w \cdot 0$
e	0	$p\omega_1$
s_1	1	$(p-2)\omega_1 + \omega_2$
$s_1 s_2$	2	$(p-5)\omega_1 + 2\omega_2$
$s_1 s_2 s_1$	3	$(p-6)\omega_1 + 2\omega_2$
$s_1 s_2 s_1 s_2$	4	$(p-6)\omega_1 + \omega_2$
$s_1 s_2 s_1 s_2 s_1$	5	$(p-5)\omega_1$

Note that each λ has the form $\lambda = a\omega_1 + b\omega_2$ for $a \geq 1$ and $0 \leq b \leq 2$. From Proposition 2.7.1, we know that for $\lambda = p\mu + w \cdot 0$,

$$\dim \mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i - \ell(w)}{2}}(u \cdot \lambda - \mu).$$

In the next section, we consider such partition functions. Since the prime p does not per se play a role in the partition function computations, we will work in a general setting.

7.2. Partitions I. Let $\lambda = a\omega_1 + b\omega_2$ for $a \geq 1$ and $0 \leq b \leq 2$. From the previous section, our goal is to make computations of

(7.2.1) $$\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot \lambda - \omega_1).$$

In particular, we will identify the least value of k for which this sum is non-zero along with the value of the sum in that case. See Proposition 7.3.3 and Proposition 7.4.4.

In order for $P_k(u \cdot \lambda - \omega_1)$ to be non-zero, $u \cdot \lambda - \omega_1$ must lie in the positive (more precisely, non-negative) root lattice. By direct computation, one finds that there are only four elements $u \in W$ for which this is true (under our assumptions on a and b above). This is summarized in the following table. The value of $u \cdot \lambda - \omega_1$ is given in the root basis.

u	$\ell(u)$	$u \cdot \lambda - \omega_1$
e	0	$(2a + 3b - 2)\alpha_1 + (a + 2b - 1)\alpha_2$
s_1	1	$(a + 3b - 3)\alpha_1 + (a + 2b - 1)\alpha_2$
s_2	1	$(2a + 3b - 2)\alpha_1 + (a + b - 2)\alpha_2$
$s_1 s_2$	2	$(a - 6)\alpha_1 + (a + b - 2)\alpha_2$

Note that in some of the cases a must be sufficiently large in order for $u \cdot \lambda - \omega_1$ to lie in the positive root lattice. Specifically, for s_1, one needs $a \geq 3$ or $b \geq 1$; for s_2, one needs $a \geq 2$ or $a \geq 1$ and $b \geq 1$ or $b \geq 2$; and for $s_1 s_2$, one needs $a \geq 6$.

7.3. Partitions II. As noted in Section 7.2, our goal is to find the least value of k such that the sum (7.2.1) is non-zero. In this section, we notice some relationships among the partition functions which will allow us to identify a range under which the sum is zero.

LEMMA 7.3.1. *Let $\lambda = a\omega_1 + b\omega_2$ with $a \geq 3$ and $0 \leq b \leq 2$. Suppose that $k \leq a + b - 2$. Then*

$$P_k(\lambda - \omega_1) = P_k(s_2 \cdot \lambda - \omega_1).$$

PROOF. Recall the table in Section 7.2, and set

$$\gamma_1 := \lambda - \omega_1 = (2a+3b-2)\alpha_1 + (a+2b-1)\alpha_2,$$
$$\gamma_2 := s_2 \cdot \lambda - \omega_1 = (2a+3b-2)\alpha_1 + (a+b-2)\alpha_2.$$

Consider a decomposition of γ_1 into k not necessarily distinct positive roots. Since $a+2b-1 = (a+b-2)+(b+1)$ and $k \leq a+b-2$, at least $b+1$ of those roots must contain $2\alpha_2$. However, the only root containing $2\alpha_2$ is $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$. Hence, any decomposition of γ_1 into k roots must contain at least $b+1$ copies of $\tilde{\alpha}$. Therefore

$$P_k(\gamma_1) = P_{k-b-1}(\gamma_1 - (b+1)(3\alpha_1+2\alpha_2)) = P_{k-b-1}((2a-5)\alpha_1 + (a-3)\alpha_2). \tag{7.3.1}$$

Now consider γ_2 and the difference between the number of α_1s and α_2s appearing. Suppose $\gamma_2 = \sum(m_i\alpha_1 + n_i\alpha_2)$ is expressed as a sum of k positive roots. Then

$$\sum(m_i - n_i) = \sum m_i - \sum n_i = (2a+3b-2) - (a+b-2) = a+2b.$$

Note that for each i, $m_i - n_i \in \{-1,0,1,2\}$. Since $k \leq a+b-2$, for at least $b+1$ values of i (in fact, at least $b+2$ values), we must have $m_i - n_i = 2$. However, the only root where that occurs is $3\alpha_1 + \alpha_2$. Hence, any decomposition of γ_2 into k roots must contain at least $b+1$ copies of $3\alpha_1+\alpha_2$. Therefore,

$$P_k(\gamma_2) = P_{k-b-1}(\gamma_2 - (b+1)(3\alpha_1+\alpha_2)) = P_k((2a-5)\alpha_1 + (a-3)\alpha_1).$$

Combining this with (7.3.1) gives the claim. $\square$

LEMMA 7.3.2. *Let $\lambda = a\omega_1 + b\omega_2$ with $a \geq 6$. Suppose that $k \leq a+b-2$. Then*

$$P_k(s_1 \cdot \lambda - \omega_1) = P_k(s_1 s_2 \cdot \lambda - \omega_1).$$

PROOF. Recall the table in Section 7.2, and set

$$\gamma_3 := s_1 \cdot \lambda - \omega_1 = (a+3b-3)\alpha_1 + (a+2b-1)\alpha_2,$$
$$\gamma_4 := s_1 s_2 \cdot \lambda - \omega_1 = (a-6)\alpha_1 + (a+b-2)\alpha_2.$$

For γ_3, the same argument as in the preceding lemma gives

$$P_k(\gamma_3) = P_{k-b-1}(\gamma_3 - (b+1)(3\alpha_1+2\alpha_2)) = P_{k-b-1}((a-6)\alpha_1 + (a-3)\alpha_2). \tag{7.3.2}$$

Now consider γ_4 and the difference between the number of α_1s and α_2s appearing. Suppose $\gamma_4 = \sum(m_i\alpha_1 + n_i\alpha_2)$ is expressed as a sum of k positive roots. Then

$$\sum(m_i - n_i) = \sum m_i - \sum n_i = (a-6) - (a+b-2) = -b-4.$$

Note that for each i, $m_i - n_i \in \{-1,0,1,2\}$. For at least $b+1$ values of i (in fact, at least $b+4$ values), we must have $m_i - n_i = -1$. However, the only root where that occurs is α_2. Hence, any decomposition of γ_4 into k roots must contain at least $b+1$ copies of α_2. Therefore

$$P_k(\gamma_4) = P_{k-b-1}(\gamma_2 - (b+1)\alpha_2) = P_{k-b-1}((a-6)\alpha_1 + (a-3)\alpha_2).$$

Combining this with (7.3.2) gives the claim. $\square$

With the two aforementioned lemmas we can now prove the following proposition.

PROPOSITION 7.3.3. *Let $\lambda = a\omega_1 + b\omega_2$ for $a \geq 1$ and $0 \leq b \leq 2$. For $k \leq a+b-2$,*

$$\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot \lambda - \omega_1) = 0.$$

PROOF. From the discussion in Section 7.2,

$$\sum_{u \in W} (-1)^{\ell(u)} P_k(u \cdot \lambda - \omega_1) = P_k(\lambda - \omega_1) - P_k(s_1 \cdot \lambda - \omega_1) - P_k(s_2 \cdot \lambda - \omega_1) + P_k(s_1 s_2 \cdot \lambda - \omega_1).$$

For $a \geq 6$, the claim follows from Lemma 7.3.1 and Lemma 7.3.2 above. For $a < 6$, one can see from the proof of Lemma 7.3.2 that the 2nd and fourth terms are zero. Hence, for $3 \leq a \leq 5$, the result follows from Lemma 7.3.1. For $1 \leq a \leq 2$, one can see from the proof of Lemma 7.3.1 that both the first and third terms vanish, and so the result follows. When a is small, the claim could also be readily verified by hand. $\square$

7.4. Partitions III. Let $\lambda = a\omega_1 + b\omega_2$ for $a \geq 1$ and $0 \leq b \leq 2$. The goal of this section is to determine

$$\sum_{u \in W} (-1)^{\ell(u)} P_{a+b-1}(u \cdot \lambda - \omega_1).$$

See Proposition 7.4.4.

From the discussion in Section 7.2 we need to consider the following weights (with notation following Section 7.3):

$$\begin{aligned}
\gamma_1 &:= \lambda - \omega_1 = (2a+3b-2)\alpha_1 + (a+2b-1)\alpha_2,\\
\gamma_2 &:= s_2 \cdot \lambda - \omega_1 = (2a+3b-2)\alpha_1 + (a+b-2)\alpha_2,\\
\gamma_3 &:= s_1 \cdot \lambda - \omega_1 = (a+3b-3)\alpha_1 + (a+2b-1)\alpha_2,\\
\gamma_4 &:= s_1 s_2 \cdot \lambda - \omega_1 = (a-6)\alpha_1 + (a+b-2)\alpha_2.
\end{aligned}$$

More precisely,

$$\sum_{u \in W} (-1)^{\ell(u)} P_{a+b-1}(u \cdot \lambda - \omega_1) = P_{a+b-1}(\gamma_1) - P_{a+b-1}(\gamma_2) - P_{a+b-1}(\gamma_3) + P_{a+b-1}(\gamma_4). \tag{7.4.1}$$

We first make some reduction observations as done in the proofs in Section 7.3. Note that when a is small, some of the statements are trivially true since both sides are zero. But we include them here (and in the following statements) for simplicity of exposition. Observe also that the right hand side is independent of the value of b.

Lemma 7.4.1. *Let λ, γ_1, γ_2, γ_3, and γ_4 be as above, and let $k = a+b-1$. Then*

(a) $P_k(\gamma_1) = P_{a-1}(2(a-1)\alpha_1 + (a-1)\alpha_2)$;
(b) $P_k(\gamma_2) = P_{a-2}((2a-5)\alpha_1 + (a-3)\alpha_2)$;
(c) $P_k(\gamma_3) = P_{a-1}((a-3)\alpha_1 + (a-1)\alpha_2)$;
(d) $P_k(\gamma_4) = P_{a-2}((a-6)\alpha_1 + (a-3)\alpha_2)$.

Proof. (a) Suppose that γ_1 is decomposed as a sum of k positive roots. Similar to the argument in Lemma 7.3.1, since $a+2b-1 = (a+b-1)+b$, at least b of those roots must (contain $2\alpha_2$ and hence) be $\tilde\alpha = 3\alpha_1 + 2\alpha_2$. Hence, $P_k(\gamma_1) = P_{k-b}(\gamma_1 - b\tilde\alpha)$, and the claim follows.

(b) Again, as in the proof of Lemma 7.3.1, since the difference in the number of α_1s and α_2s appearing in γ_2 is $a+2b$, if γ_2 is expressed as k roots, then at least $b+1$ of them must be $3\alpha_1+\alpha_2$. Hence, $P_k(\gamma_2) = P_{k-b-1}(\gamma_2 - (b+1)(3\alpha_1+\alpha_2))$, and the claim follows.

(c) As in part (a), we must have $P_k(\gamma_3) = P_{k-b}(\gamma_3 - b\tilde\alpha)$, and the claim follows.

(d) As in part (b), similar to the proof of Lemma 7.3.2, we consider the difference in the number of α_1s and α_2s appearing in γ_4. Since this number is $-b-4$, we can in particular assume that if γ_4 is decomposed into k roots, then at least $b+1$ of them are α_2. Hence, $P_k(\gamma_4) = P_{k-b-1}(\gamma_4 - (b+1)\alpha_2)$, and the claim follows. □

With the aid of Lemma 7.4.1, we now observe that there are some relationships among the $P_k(\gamma_i)$. To this end, we introduce a bit of notation. Consider an arbitrary integer $k \geq 0$ and weight $\gamma = c\alpha_1 + d\alpha_2$ for $c, d \geq 0$. Any decomposition of γ into a sum of k positive roots is of one of two types: either the sum contains at least one copy of $\tilde\alpha$ or it does not contain any copies of $\tilde\alpha$. Correspondingly, let $P_{k,\tilde\alpha}(\gamma)$ and $P_{k,\not\tilde\alpha}(\gamma)$ denote the number of such root sums. Then $P_k(\gamma) = P_{k,\tilde\alpha}(\gamma) + P_{k,\not\tilde\alpha}(\gamma)$. Observe that

$$P_{k,\tilde\alpha}(\gamma) = P_{k-1}(\gamma - \tilde\alpha). \tag{7.4.2}$$

Lemma 7.4.2. *Let λ, γ_1, γ_2, γ_3, and γ_4 be as above, and let $k = a+b-1$. Then*

(a) $P_k(\gamma_1) = P_k(\gamma_2) + P_{a-1,\not\tilde\alpha}(2(a-1)\alpha_1 + (a-1)\alpha_2)$;
(b) $P_k(\gamma_3) = P_k(\gamma_4) + P_{a-1,\not\tilde\alpha}((a-3)\alpha_1 + (a-1)\alpha_2)$.

Proof. (a) We have

$$\begin{aligned}
P_k(\gamma_1) &= P_{a-1}(2(a-1)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by Lemma 7.4.1(a))}\\
&= P_{a-1,\tilde\alpha}(2(a-1)\alpha_1 + (a-1)\alpha_2) + P_{a-1,\not\tilde\alpha}(2(a-1)\alpha_1 + (a-1)\alpha_2)\\
&= P_{a-2}((2a-5)\alpha_1 + (a-3)\alpha_2) + P_{a-1,\not\tilde\alpha}(2(a-1)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by (7.4.2))}\\
&= P_k(\gamma_2) + P_{a-1,\not\tilde\alpha}(2(a-1)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by Lemma 7.4.1(b))}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
P_k(\gamma_3) &= P_{a-1}((a-3)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by Lemma 7.4.1(c))}\\
&= P_{a-1,\tilde{\alpha}}((a-3)\alpha_1 + (a-1)\alpha_2) + P_{a-1,\not\tilde{\alpha}}((a-3)\alpha_1 + (a-1)\alpha_2)\\
&= P_{a-2}((a-6)\alpha_1 + (a-3)\alpha_2) + P_{a-1,\not\tilde{\alpha}}((a-3)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by (7.4.2))}\\
&= P_k(\gamma_4) + P_{a-1,\not\tilde{\alpha}}((a-3)\alpha_1 + (a-1)\alpha_2) \qquad \text{(by Lemma 7.4.1(d)).}
\end{aligned}$$

□

From (7.4.1), Lemma 7.4.1 and Lemma 7.4.2, we see that

$$(7.4.3)\quad \sum_{u\in W}(-1)^{\ell(u)}P_{a+b-1}(u\cdot\lambda - \omega_1) = P_{a-1,\not\tilde{\alpha}}(2(a-1)\alpha_1 + (a-1)\alpha_2) - P_{a-1,\not\tilde{\alpha}}((a-3)\alpha_1 + (a-1)\alpha_2).$$

LEMMA 7.4.3. *Let $c \geq 0$. Then*

$$P_{c,\not\tilde{\alpha}}(2c\alpha_1 + c\alpha_2) - P_{c,\not\tilde{\alpha}}((c-2)\alpha_1 + c\alpha_2) = \left\lceil \frac{c+1}{3} \right\rceil,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x.

PROOF. Let $\eta_1 := 2c\alpha_1 + c\alpha_2$ and $\eta_2 := (c-2)\alpha_1 + c\alpha_2$. Observe first that if $c < 2$, then $P_{c,\not\tilde{\alpha}}(\eta_2) = 0$. On the other hand, we have $P_{0,\not\tilde{\alpha}}(0) = 1$ and $P_{1,\not\tilde{\alpha}}(2\alpha_1 + \alpha_2) = 1$, and so the claim holds for $c < 2$. Assume for the remainder of the proof that $c \geq 2$.

Observe that if η_i is expressed as a sum of c positive roots, none of which are $\tilde{\alpha}$, then each root is necessarily of the form $a\alpha_1 + \alpha_2$ for $a \in \{0,1,2,3\}$. So the question of possible decompositions involves looking only at the coefficients of α_1. For nonnegative integers m, n, let $P_m(n)$ denote the number of ways that n can be expressed as a sum of m integers

$$n = n_1 + n_2 + \cdots + n_m$$

where $n_i \in \{0,1,2,3\}$. With this notation, $P_{c,\not\tilde{\alpha}}(\eta_1) = P_c(2c)$, $P_{c,\not\tilde{\alpha}}(\eta_2) = P_c(c-2)$, and our goal is to compute $P_c(2c) - P_c(c-2)$ (when $c \geq 2$).

For m, n as above, let $S_m(n)$ denote the set of such partitions of n into m integers. We first show that there is an injection $\varphi : S_c(c-2) \to S_c(2c)$. Let $\tau \in S_c(c-2)$. Say

$$\tau : c-2 = \tau_1 + \tau_2 + \cdots + \tau_c,$$

where $\tau_i \in \{0,1,2,3\}$. Let s denote the number of τ_is which equal 3. The remaining $c-s$ values must sum to $c-2-3s$, and hence at most $c-2-3s$ of those terms can be non-zero. In other words, at least $(c-s)-(c-2-3s) = 2s+2$ of the remaining terms are zero. That is, we may assume that τ has the form:

$$c-2 = \underbrace{3+\cdots+3}_{s \text{ times}} + \underbrace{0+\cdots+0}_{(2s+2) \text{ times}} + \tau_{3s+3} + \cdots + \tau_c,$$

where, for $(3s+3) \leq i \leq c$, $\tau_i \in \{0,1,2\}$. Let $\varphi(\tau)$ be the partition:

$$2c = \underbrace{3+\cdots+3}_{2s \text{ times}} + 2 + 2 + \underbrace{0+\cdots+0}_{s \text{ times}} + (\tau_{3s+3}+1) + (\tau_{3s+4}+1) + \cdots + (\tau_c+1).$$

In words, the map φ leaves the initial s copies of 3 fixed, sends s of the zeros to 3, sends two of the zeros to 2, leaves the other s zeros fixed, and adds one to the unknown integers at the end. Note that those unknown integers are each at most 2, so adding one is allowable. One can also readily check that the new sum does indeed add up to $2c$. It is clear that φ is an injection, but we will explicitly construct an inverse below.

Observe that the resulting partition of $2c$ contains 2 at least twice. We claim that the image of φ is in fact precisely the subset $X \subset S_c(2c)$ consisting of those partitions where 2 appears two or more times. Indeed, we can define a function $\psi : X \to S_c(c-2)$ as follows. Let $\xi \in X$ and s denote the number of times that zero appears in ξ. The remaining $c-s$ values in ξ must sum to $2c$. We know that at least two of those have value 2. The remaining $c-s-2$ terms must sum to $2c-4$. Since $2(c-s-2) = 2c-4-2s$, at least $2s$ of those terms must have value 3. In other words, ξ has the form:

$$2c = \underbrace{0+\cdots+0}_{s \text{ times}} + 2 + 2 + \underbrace{3+\cdots+3}_{2s \text{ times}} + \xi_{3s+3} + \cdots + \xi_c,$$

where (for $(3s+3) \leq i \leq c$) $\xi_i \in \{1,2,3\}$. Let $\psi(\xi)$ be the partition:

$$c-2 = \underbrace{0+\cdots+0}_{(2s+2)\text{ times}} + \underbrace{3+\cdots+3}_{s\text{ times}} + (\xi_{3s+3}-1) + (\xi_{3s+4}-1) + \cdots + (\xi_c - 1).$$

In words, the map ψ leaves the zeros fixed, sends the two 2s to zero, sends s copies of 3 to zero, leaves the other s copies of 3 fixed, and subtracts one from each of the remaining integers. Clearly ψ is an inverse to ϕ. Hence, $P_c(c-2) = |X|$.

It remains to compute $P_c(2c) - |X|$. That is, we need to count the number of partitions

$$2c = n_1 + n_2 + \cdots + n_c,$$

where $n_i \in \{0,1,2,3\}$ but for which at most one value of $n_i = 2$. Write $c = 3m+t$ for $m \geq 0$ and $t < 3$. Then it is a straightforward (but somewhat lengthy) computation to show that the number of such partitions is $m+1$. This is left to the interested reader. The lemma follows. □

Applying Lemma 7.4.3 with $c = a-1$, we obtain the following from (7.4.3).

PROPOSITION 7.4.4. *Let $\lambda = a\omega_1 + b\omega_2$ for $a \geq 1$ and $0 \leq b \leq 2$. Then*

$$\sum_{u\in W} (-1)^{\ell(u)} P_{a+b-1}(u\cdot\lambda - \omega_1) = \left\lceil \frac{a}{3} \right\rceil.$$

7.5. Vanishing Ranges. Suppose that $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $\lambda \in X(T)_+$ and $i > 0$. From the discussion in Section 7.1, we know that if $i < 2p-3$, then λ must be of the form $\lambda = p\omega_1 + w\cdot 0$, and more precisely, that it must be one of the weights listed in Table 7.1. For each such λ, from Proposition 7.3.3 and Proposition 7.4.4, we can identify the least value of k such that

$$\sum_{u\in W} (-1)^{\ell(u)} P_k(u\cdot\lambda - \omega_1) \neq 0,$$

and moreover, identify the value of the sum. Further, from Proposition 2.7.1 (with $k = (i-\ell(w))/2)$), we can then identify the least non-negative i with $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ along with the dimension of the cohomology group. This information is summarized in the following table. Here k and i are minimum possible values, and dim gives the dimension of the cohomology group (equivalently the value of (7.2.1)).

w	$\ell(w)$	$\lambda = p\omega_1 + w\cdot 0$	k	i	dim
e	0	$p\omega_1$	$p-1$	$2p-2$	$\lceil \frac{p}{3} \rceil$
s_1	1	$(p-2)\omega_1 + \omega_2$	$p-2$	$2p-3$	$\lceil \frac{p}{3} \rceil - 1$
s_1s_2	2	$(p-5)\omega_1 + 2\omega_2$	$p-4$	$2p-6$	$\lceil \frac{p}{3} \rceil - 2$
$s_1s_2s_1$	3	$(p-6)\omega_1 + 2\omega_2$	$p-5$	$2p-7$	$\lceil \frac{p}{3} \rceil - 2$
$s_1s_2s_1s_2$	4	$(p-6)\omega_1 + \omega_2$	$p-6$	$2p-8$	$\lceil \frac{p}{3} \rceil - 2$
$s_1s_2s_1s_2s_1$	5	$(p-5)\omega_1$	$p-6$	$2p-7$	$\lceil \frac{p}{3} \rceil - 2$

THEOREM 7.5.1. *Suppose Φ is of type G_2 and $p \geq 7$.*

(a) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ *for $i < 2p-8$.*

(b) $\dim \mathrm{H}^{2p-8}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = \begin{cases} \lceil \frac{p}{3} \rceil - 2 & \text{if } \lambda = (p-6)\omega_1 + \omega_2 \\ 0 & \text{else.} \end{cases}$

(c) $\dim \mathrm{H}^{2p-7}(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = \begin{cases} \lceil \frac{p}{3} \rceil - 2 & \text{if } \lambda = (p-5)\omega_1 \text{ or } (p-6)\omega_1 + 2\omega_2 \\ 0 & \text{else.} \end{cases}$

(d) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-8$.

Proof. Part (a) follows from Proposition 2.7.1 and Proposition 7.3.3. Parts (b) and (c) follow from the preceding table and discussion. Part (d) follows from part (a) and Proposition 2.4.1. □

One would like to apply Theorem 2.5.1 to conclude that $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) \neq 0$. However, the weight $(p-5)\omega_1$ is less than and linked to the weight $(p-6)\omega_1+\omega_2$ and so the Theorem is not applicable. The non-zero cohomology from the weight $(p-5)\omega_1$ in degree $2p-7$ could "cancel" some or all of the cohomology coming from the weight $(p-6)\omega_1+\omega_2$. We refer the interested reader to [BNP, Section 2.7] for discussion of this interplay.

In a similar manner, cohomology in degree $2p-6$ coming from the weight $(p-6)\omega_1+\omega_2$ could cancel that in degree $2p-7$ coming from the weight $(p-6)\omega_1+2\omega_2$. So it is not even possible to conclude that $\mathrm{H}^{2p-7}(G(\mathbb{F}_p), k) \neq 0$. In summary, alternate methods are needed to determine the precise vanishing bound.

8. Type F_4

Assume throughout this section that Φ is of type F_4 and that $p > h = 12$ (so $p \geq 13$). Following the strategy laid out in Section 2, our goal is to find the least $i > 0$ such that $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $\lambda \in X(T)_+$.

8.1. Restrictions. Suppose that $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i > 0$ and $\lambda = p\mu + w\cdot 0$ with $\mu \in X(T)_+$ and $w \in W$. From Proposition 2.8.1, $i \geq (p-1)\langle \mu, \tilde{\alpha}^\vee\rangle - 1$. For $1 \leq i \leq 3$, we have $\langle \omega_i, \tilde{\alpha}^\vee\rangle \geq 2$, while $\langle \omega_4, \tilde{\alpha}^\vee\rangle = 1$. Therefore, unless $\mu = \omega_4 = \alpha_0$, we have $\langle \mu, \tilde{\alpha}^\vee\rangle \geq 2$ and $i \geq 2p-3$.

Suppose now that $\lambda = p\omega_4 + w\cdot 0$ for some $w \in W$. With the aid of MAGMA, one can identify all w for which λ is in fact dominant. From Proposition 2.8.1(a), with $\sigma = \alpha_0$ and $\lambda = p\omega_4 + w\cdot 0$, since $\langle \omega_4, \alpha_0^\vee\rangle = 2$, we have

$$i \geq 2(p-1) + \ell(w) + \langle w\cdot 0, \alpha_0^\vee\rangle. \tag{8.1.1}$$

By checking all possible cases, one finds that $\ell(w) + \langle w\cdot 0, \alpha_0^\vee\rangle \geq -7$. Combining this with (8.1.1), we conclude that $i \geq 2p-9$. From Proposition 2.4.1, we get the following.

Theorem 8.1.1. *Suppose Φ is of type F_4 and $p \geq 13$. Let $\lambda \in X(T)_+$. Then*

(a) $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-9$;

(b) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-9$.

8.2. Based on the preceding discussion, the weights which could give $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $i \leq 2p-7$ are summarized in the following table.

$\lambda = p\omega_4 + w\cdot 0$	$\ell(w)$	$\langle w\cdot 0, \alpha_0^\vee\rangle$	i
$(p-12)\omega_4+\omega_2$	13	-20	$2p-9$
$(p-12)\omega_4+\omega_3$	14	-21	$2p-9$
$(p-11)\omega_4$	15	-22	$2p-9$
$(p-11)\omega_4+3\omega_1$	10	-16	$2p-8$
$(p-12)\omega_4+2\omega_1+\omega_3$	11	-17	$2p-8$
$(p-12)\omega_4+\omega_1+\omega_2$	12	-18	$2p-8$
$(p-11)\omega_4+2\omega_1+\omega_2$	9	-14	$2p-7$
$(p-12)\omega_4+\omega_1+\omega_2+\omega_3$	10	-15	$2p-7$
$(p-12)\omega_4+2\omega_2$	11	-16	$2p-7$

As seen in Section 2.7, $\lambda - \omega_4$ must be a weight of $S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*)$, and hence i is congruent to $\ell(w)$ mod 2. It follows that some of the above degree bounds are even higher. For example, consider $\lambda = (p-12)\omega_4 + \omega_3 =$

$p\omega_4 + w\cdot 0$. Since $\ell(w) = 14$ but $2p-9$ is odd, the least value i could take would be $2p-8$. A similar situation holds for $\lambda = (p-12)\omega_4 + 2\omega_1 + \omega_3$ and $\lambda = (p-12)\omega_4 + \omega_1 + \omega_2 + \omega_3$. Similarly, for the other weights in the above list, if the cohomology vanishes in the degree i listed, then the next possible non-vanishing degree is $i+2$. We summarize this in the following lemma.

LEMMA 8.2.1. *Suppose Φ is of type F_4, $p \geq 13$ and $\lambda \in X(T)_+$. Suppose that* $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) \neq 0$. *Then*

(a) $i \geq 2p-9$;
(b) *if* $i = 2p-9$, *then* $\lambda = (p-12)\omega_4 + \omega_2$ *or* $(p-11)\omega_4$;
(c) *if* $i = 2p-8$, *then* $\lambda = (p-12)\omega_4 + \omega_3$, $(p-11)\omega_4 + 3\omega_1$, *or* $(p-12)\omega_4 + \omega_1 + \omega_2$;
(d) *if* $i = 2p-7$, *then* $\lambda = (p-12)\omega_4 + \omega_2$, $(p-11)\omega_4$, $(p-12)\omega_4 + 2\omega_1 + \omega_3$, $(p-11)\omega_4 + 2\omega_1 + \omega_2$, *or* $(p-12)\omega_4 + 2\omega_2$.

8.3. Conjectures. In principle, one could use Proposition 2.7.1 to compute the dimension of

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$$

in terms of partition functions for the weights in Lemma 8.2.1. For small p, one can use MAGMA to make this computation. For $p = 13, 17$, or 19, one finds that the two candidates in degree $2p-9$ have zero cohomology. They do give cohomology in degree $2p-7$. And in degree $2p-8$, the only one weight (of the three) which has cohomology is $(p-12)\omega_4 + \omega_3$. We make the following

CONJECTURE 8.3.1. *Suppose that Φ is of type F_4, $p \geq 13$, and $\lambda = p\mu + w\cdot 0 \in X(T)_+$. Then*

(a) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ *for* $0 < i < 2p-8$;
(b) $\mathrm{H}^{2p-8}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ *for* $\lambda = (p-12)\omega_4 + \omega_3$.

If part (a) of the conjecture holds, then $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p-8$ thus improving upon Theorem 8.1.1. However, even if part (b) of the conjecture also holds, it does not necessarily follow that $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) \neq 0$. Analogous to the situation for type G_2 (cf. Section 7.5), cohomology in degree $2p-7$ from the weight $(p-11)\omega_4$ could cancel out the cohomology in degree $2p-8$ from the weight $(p-12)\omega_4 + \omega_3$.

Conjecture 8.3.1 is a special case of a more general conjecture on partition functions (known to hold for small values of m). Conjecture 8.3.1(a) would follow from parts (a) and (b) while Conjecture 8.3.1(b) would follow from part (c).

CONJECTURE 8.3.2. *Suppose that Φ is of type F_4 and $m \geq 1$. Then*

(a) $\displaystyle\sum_{u\in W} (-1)^{\ell(u)} P_{m+1}(u\cdot(m\omega_4 + \omega_2) - \omega_4) = 0$;

(b) $\displaystyle\sum_{u\in W} (-1)^{\ell(u)} P_{m-1}(u\cdot(m\omega_4) - \omega_4) = \begin{cases} 0 & \textit{if } m \textit{ is even,} \\ 1 & \textit{if } m \textit{ is odd;} \end{cases}$

(c) $\displaystyle\sum_{u\in W} (-1)^{\ell(u)} P_{m+1}(u\cdot(m\omega_4 + \omega_3) - \omega_4) = 1$.

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